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The Explosive Growth and Rapid Contraction of an Overlapping Generations Economy

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Abstract

This paper examines the long-term consequences of population decline in an endogenous growth framework with stochastic labor-augmenting technological progress. The key factor in fertility decisions is the cost of child-rearing, which is modeled as a convex function of labor productivity. As technology grows, the costs of raising children (including childbearing, childcare, and educational investments) increase disproportionately. While the economy may experience rapid growth in the early stages of development, it is likely to face sharp contractions in both population and innovation as child-rearing costs outpace the incentives for having children. We show that consistent population decay is almost inevitable. Nevertheless, a pronatalist policy can increase the likelihood of achieving high long-run labor productivity and living standards for future generations, although some short-run welfare deficits are to be expected.

Keywords: stochastic labor-augmenting growth, overlapping generations, endogenous fertility

JEL Classification: E13, J13, J14, J22, J24, O11

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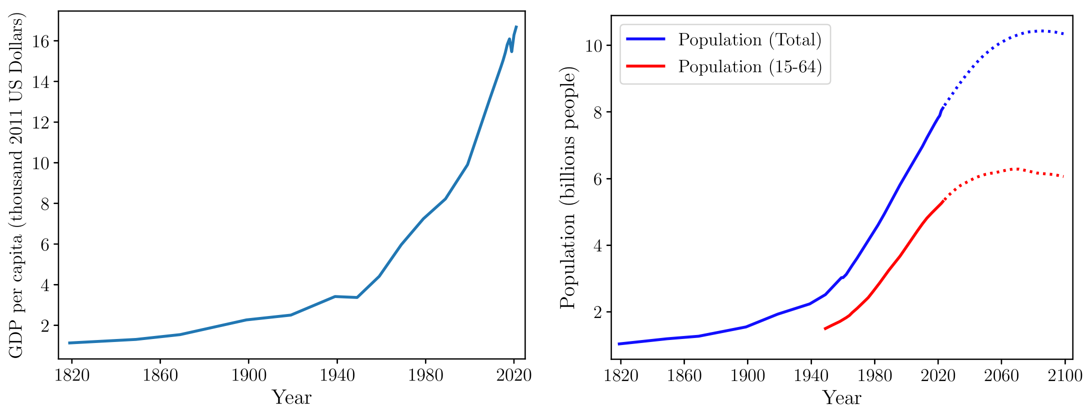
1 Introduction

Throughout the majority of human history, the population has increased exponentially. Based on the available data dating back to the beginning of the first industrial revolution in 1820, Figure 1 shows the evolution of GDP per capita and population. Compared to two centuries ago, the world GDP per capita and population have expanded by seventeen-fold and eight-fold, respectively. The growing population has brought more workers, talents, ideas, and higher demands for technological innovation (Galor and Weil, 2000). As a result, human survival rates, income, and quality of life have improved remarkably. However, many estimations suggest that the era of an ever-growing population may be coming to an end.

> Populations are already declining in about two dozen states around the world; by 2050 the number will have climbed to three dozen. Some of the richest places on earth are shedding people every year: Japan, Korea, Spain, Italy, much of Eastern Europe. “We are a dying country,” Italy’s health minister, Beatrice Lorenzin, lamented in 2015.
>
> Bricker and Ibbitson (2019, p.5)



(a) World Average GDP per capita.

(b) World Population.

Figure 1: World Average GDP per capita and Population, with the dotted lines representing projections. Data source: Maddison Project (2023), UN’s World Population Prospects 2024.

Figure 1b shows medium scenario projections from the UN’s World Population Prospects 2024. They predict that the world population will peak around the mid-2080s at 10.3 billion and then begin to decline in absolute number. Perhaps more alarmingly, the decline of the working-age population may happen even sooner, within the next few decades. The UN’s predictions are not just estimates; they are nearly bulletproof. History shows that most of their predictions differed from the latest estimates by just 1% to 2%, and none differed by more than 5% (Keilman, 2001). In other words, the scenario of a population contraction is highly probable. In that case, what would happen to economic growth and the population in the long run?

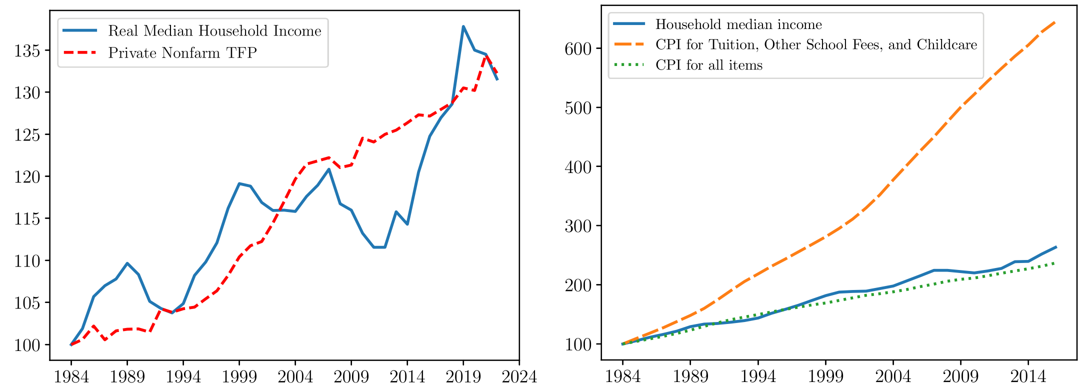
This paper addresses the issue by developing an overlapping generations (OLG) model à la Diamond (1965), incorporating endogenous fertility and labor-augmenting technological progress. The decision of parents on the number of children to raise has an important implication on growth, as innovation is positively related to the size of the population (Romer, 1990; Segerstrom et al., 1990; Grossman and Helpman, 1991a). A larger population increases the likelihood of exceptional talents emerging, which can lead to breakthroughs in technological progress. Empirical evidence suggests that societies with a high population are more likely to attain higher technology levels (Kremer, 1993).

The altruistic behavior of parents follows the "warm glow" approach, similar to Galor and Weil (1996) and de la Croix and Doepke (2003), in which parents derive utility from having children. However, bearing children comes with a cost. In this model, we treat child-rearing costs explicitly as financial expenses. While many existing models assume that child-rearing costs are fixed over time (Blackburn and Cipriani, 1998; Fanti and Gori, 2014), or proportional to income, the model considered here allows these costs to evolve dynamically. To be specific, we assume that when innovation occurs, it raises the child-rearing costs more than the gains in income. While we do not provide a micro-foundation to this phenomenon, we provide some empirical evidence from other studies that have shown such a relationship.

Figure 2a shows a high correlation between labor-augmenting technological progress and households' real median income, with data since 1984 for the US. The general trends suggest that technological progress would lead to a one-for-one increase in income. On the other hand, figure 2b shows that child-rearing costs have been rising faster than median income and all other items. Since 1984, while the price index for tuition, childcare, and other school fees has risen by more than fivefold, the median household income has only doubled during the same period. That is, child-rearing costs are getting more unaffordable for many American families. These facts suggest that modeling child-rearing costs as a fixed amount of goods cost or a constant fraction of income might not be suitable to capture this relationship.

Kubota (2020) documents a sharp increase in childcare prices in the United States since the late 1990s that substantially outpaced general price inflation. On the international front, the OECD reports that families in many developed countries spend upward of 10–30% of household income on childcare (OECD, 2020), and these shares have been trending upward, consistent with costs rising faster than incomes. Such a trend can also be observed in many developed countries (Shine, 2023).

To capture this relationship, this model assumes that child-rearing cost is a convex function of efficient labor (which can also be viewed as labor productivity in the model). First, higher labor productivity leads to higher labor compensation and service prices, which, in turn, raise child-rearing costs, such as childbearing, childcare, and college tuition. One plausible channel is that the cost of children nowadays is highly elastic to human capital components, such as education and health (Ogawa et al., 2009). Its soaring price can be ex-



(a) TFP and income (b) Child-rearing costs

Figure 2: Income and Child-rearing Costs in the US from 1984 to 2016 with 1984=100. Data from FRED.

plained by the changes in service prices of these sectors due to technological progress [1]. College costs (which take up a large portion of parents' expenses on children) have been rising faster than wages (Vedder, 2004). Archibald and Feldman (2011) argue that the rapid rise in college tuition can be driven by the nature of technological progress [2]. Although technological progress can lower the cost of making the same old thing in other industries, education tends to experience the kind of technological progress that raises quality and cost. In other words, as the standards of labor productivity change, colleges and universities have increased the cost of equipping their students with state-of-the-art education and cutting-edge research. Since child-rearing also includes pre-college childcare, which has also become more expensive related to parents' income (Herbst, 2023). Another possible channel relates to Baumol's cost disease, in which service sectors such as education, healthcare, and childcare experience above-average price increases due to their labor-intensive nature and wage increases that exceed productivity growth (Hartwig, 2008). Since there are many potential causes of why child-rearing is becoming less affordable, we focus on its implications and leave a comprehensive microfoundations for future research.

The model is built in reference to a large body of work on economic growth, including Jones (1995), Romer (1990), Aghion and Howitt (1992), and Grossman and Helpman (1991b), which has emphasized the central role of population size and fertility. The ongoing decline in fertility rates, and in some cases outright

[1] According to Sallie Mae (2024), about 77% of American families rely on their parents' income and savings for college expenses. For more empirical evidence, we refer to Hotz et al. (2023). On a global scale, the vast majority of parents (87%) are funding (either fully or partially) their children's education. Among those with children in university or college, 85% of them are supporting their children financially (HSBC, 2017). Therefore, it is natural to assume that tuition costs (which correlate with labor productivity) account for a large part of parents' child-rearing expenses.

[2] Archibald and Feldman (2011) also point out other causes of the sharp rise in college costs, such as the Baumol cost disease (Baumol and Blackman, 1995) and the lagging of technological progress in service-providing industries.

population decline, has sparked debate on these issues. Yet research that examines the consequences of absolute population decline remains scarce and has only recently begun to take shape.

For instance, Jones (2022a) considers a model that permits population decline in an R&D endogenous growth framework. Sasaki and Hoshida (2017) also considers the effect of population decline in a semi-endogenous growth framework. Daitoh (2020) examines child-rearing costs in a related context. Futagami and Konishi (2019) analyze fertility dynamics in an OLG model with R&D-based growth, demonstrating a reversal in the fertility–income relationship at high development levels. Still, the link to endogenous fertility and convex child-rearing costs has not been fully addressed. Jones (2022a)'s model identifies two key steady states: an "empty planet" state characterized by an asymptotic zero population and no new ideas in the long run, and the other called "expanding cosmos" with perpetual population growth and ongoing innovation. However, the mechanism for negative population growth arises from preference parameters, which limits its application to studying demographic transitions.

In contrast, our model connects population dynamics to labor productivity. Technological improvements increase both household incomes and child-rearing costs, creating a tension where the income effect encourages childbearing, but the rising costs for future generations discourage it. This dynamic determines whether fertility rates fall below replacement levels. Our approach also allows the model to capture both historical population growth patterns and the possibility of future negative growth. Moreover, it suggests that the "empty planet" scenario might be the only equilibrium in the long run. While it may not be possible to reverse this trend, a pronatalist policy can still benefit society by improving the potential long-run labor productivity and living standards of the distant generations.

The rest of the paper is organized as follows. Section 2 introduces the structure of the model. In section 3, we derive the competitive equilibrium and the law of motion for key variables. Section 4 then shows the long-run labor productivity and population, where the zero-population scenario is shown to be almost inevitable. Section 6 then studies the effectiveness of a child-rearing subsidy policy. Finally, section 7 concludes.

2 Model

A new generation of three-period-lived agents is born at the beginning of each period. In the first period of life (young), the agents make no economic decisions. In the second period (middle-aged), they consume, work, give birth, and raise children. In the third period (old age), the agents consume all the goods they saved in the second period and die.

2.1 Production and technological progress

The final good is competitively produced by a representative firm from capital and labor using the following production technology:

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}, \tag{1}$$

where Y_t, K_t, A_t, N_t are the final output, capital, labor-augmenting technology, and the number of workers available for work in period t. In this formulation, A_t can be interpreted as the number of "efficiency units of labor" that each worker supplies. A worker in period t can supply A_t units of labor in terms of productive capacity. Hence, $A_t N_t$ represents the total effective labor supply in the economy. This interpretation is standard in growth theory with Harrod-neutral (labor-augmenting) technological progress (Barro and Sala-i Martin, 2004; Romer, 2012).

In every period, either state 0 or state 1 is realized where state 0 indicates no technological progress while state 1 indicates the existence of a breakthrough in technology. The level of technology A_t evolves according to the following:

$$A_t = \begin{cases} A_{t-1} & \text{if state 0 is realized,} \\ aA_{t-1} & \text{if state 1 is realized,} \end{cases} \tag{2}$$

where a is a constant larger than 1. There is no technological obsolescence. As a result, even when state 0 is realized, the level of labor-augmenting technology does not decrease. Eq.(2) implies that the effective labor supply in period t is given by

$$A_t N_t = \begin{cases} A_{t-1} N_t & \text{if state 0 is realized,} \\ aA_{t-1} N_t & \text{if state 1 is realized.} \end{cases} \tag{3}$$

Since a period in the model is long (approximately 25-30 years, corresponding to one generation), capital is assumed to fully depreciate after one period, which is a standard simplification in OLG models (de la Croix and Michel, 2002).[3] Using (3), we can express the factor market equilibria in period t. When state 0 is realized in period t, we have:

$$r_t^0 = \alpha \left(\frac{K_t}{A_{t-1} N_t} \right)^{\alpha - 1}, \tag{4}$$

$$w_t^0 = (1-\alpha) \left(\frac{K_t}{A_{t-1} N_t} \right)^{\alpha}, \tag{5}$$

[3] With a conventional annual depreciation rate of $\delta = 0.1$, the fraction of capital remaining after 25 years is $(1-\delta)^{25} \approx 0.07$, making full depreciation a reasonable approximation for our generational framework. Moreover, the no-depreciation assumption on technology reflects the nature of knowledge: once discovered, a technological breakthrough persists in the form of accumulated knowledge and human capital, even if subsequent innovations build upon it. Physical capital, by contrast, is subject to wear and tear over each generation's lifespan.

where the superscript 0 indicates that the price was established in state 0. On the other hand, when state 1 is realized, they can be expressed as

$$r_t^1 = \alpha \left(\frac{K_t}{aA_{t-1}N_t} \right)^{\alpha-1}, \tag{6}$$

$$w_t^1 = (1-\alpha) \left(\frac{K_t}{aA_{t-1}N_t} \right)^{\alpha}. \tag{7}$$

where the superscript 1 indicates that the price was established in state 1. Note that both w_t^0 and w_t^1 are the wage rates per efficiency unit of labor. Under the interpretation that each worker supplies A_t efficiency units of labor, the labor income per worker is $w_t^0 A_t$ and $w_t^1 A_t$ in state 0 and state 1, respectively. This formulation implies that technological progress increases worker compensation through the quantity of efficiency units supplied (A_t), while the price per efficiency unit (w_t) is determined by factor market equilibrium.

In period t, state 0 is realized with the probability of $1 - p_t$, and state 1 is realized with the probability of p_t. Following the endogenous growth literature, we assume that the function of p_t depends on population and satisfies

$$p_t = f(N_t) \tag{8}$$

where

$$\partial p_t / \partial N_t > 0, \quad \lim_{N_t \to 0} p_t = 0, \quad \lim_{N_t \to +\infty} p_t = 1. \tag{9}$$

The idea behind this formulation is that the larger the population, the greater the probability that technological progress occurs. As shown in Chu and Cozzi (2019), the scale effect is an important factor in innovation. [4] . It may be due to a larger talent pool of innovators (Galor, 2011), the need to increase innovation to accommodate more people with limited natural resources (Kremer, 1993), and market-sized effects in which the growth rate of technologies increases with the amount of labor that uses them (Acemoglu, 2002).

Another key assumption is that there is no technological obsolescence: when innovation fails to occur (state 0), technology remains at its previous level rather than depreciating. This assumption is motivated by the observation that at the frontier of technological development, major technological advances tend to be cumulative and persistent (Mokyr, 2002). While some technologies may become obsolete, the underlying knowledge and human capital typically persist, preventing a decline in the aggregate technology level.

[4] Due to the lack of microfoundation on R&D activities, Eq. (8) does not depend on the fertility rate (or population growth), as commonly appears in the semi-endogenous growth literature (Jones, 1995, 2022b; Cozzi, 2017).

2.2 Individuals

2.2.1 Child-rearing cost

For each child, parents' expenditure on childcare is assumed to take the following form

$$z_t = \bar{z} A_t^{\phi}, \tag{10}$$

where $\bar{z}$ is a positive constant, and ϕ is the elasticity of child-rearing cost with respect to the efficiency units of labor. To capture the fact that child-rearing costs rise faster than technological progress, we assume that $\phi > 1$ [5] .

This formulation represents a simple relationship between child-rearing costs and the labor-augmenting technology. When labor productivity improves, wages increase by a factor of A_t, while child-rearing cost gets more expensive as they increase by a factor of A_t^{ϕ}.

2.2.2 State 0

Let $c_t^0, s_t^0, n_t^0, d_{t+1}^0$ denote consumption, savings in the middle age, the number of children, and consumption (dissave) in the old age, respectively. When state 0 is realized in period t, all middle-aged agents in that period attain the following preferences

$$U_t^0 = \log c_t^0 + \gamma \log n_t^0 + \beta E_t \log d_{t+1}^0, \tag{11}$$

where γ and β are positive constants governing the tastes of children and future consumption. They maximize (11) subject to the following constraints:

$$w_t^0 A_t = c_t^0 + s_t^0 + z_t n_t^0, \tag{12}$$

$$d_{t+1}^0 = E_t(r_{t+1}) s_t^0 = \begin{cases} r_{t+1}^0 s_t^0 & \text{if state 0 is realized in period } t+1, \\ r_{t+1}^1 s_t^0 & \text{if state 1 is realized in period } t+1, \end{cases} \tag{13}$$

Using Eqs.(10)-(13), the maximization problem can be expressed as

$$\max_{n_t^0, s_t^0} \log(w_t^0 A_t - s_t^0 - z_t n_t^0) + \gamma \log n_t^0 + \beta \left[(1 - p_{t+1}) \log r_{t+1}^0 s_t^0 + p_{t+1} \log r_{t+1}^1 s_t^0 \right].$$

The first-order conditions are

$$\frac{-z_t}{w_t^0 A_t - s_t^0 - z_t n_t^0} + \frac{\gamma}{n_t^0} = 0,$$
$$\frac{-1}{w_t^0 A_t - s_t^0 - z_t n_t^0} + \frac{\beta}{s_t^0} = 0.$$

[5] An alternative formulation would be $z_t = \bar{z} A_t^{\phi} w_t$, which would make child-rearing costs proportional to the wage rate in addition to technology. However, as shown in Appendix C, following such a formula implies that fertility will never fall, which is unrealistic.

We obtain:

$$n_t^0 = \frac{\gamma}{1+\beta+\gamma} \cdot \frac{w_t^0 A_t}{z_t}, \tag{14}$$

$$s_t^0 = \frac{\beta}{1+\beta+\gamma} w_t^0 A_t. \tag{15}$$

The laws of motion for the capital stock and population are

$$K_{t+1} = s_t^0 N_t, \tag{16}$$

$$N_{t+1} = n_t^0 N_t. \tag{17}$$

Eqs.(10) and (14)–(17) jointly imply:

$$K_{t+1}/N_{t+1} = \beta z_t/\gamma = \beta \bar{z} A_t^{\phi}/\gamma. \tag{18}$$

2.2.3 State 1

When state 1 is realized in period t, the middle-aged agents' utility can be represented by:

$$U_t^1 = \log c_t^1 + \gamma \log n_t^1 + \beta E_t \log d_{t+1}^1, \tag{19}$$

with c_t^1, s_t^1, n_t^1, and d_{t+1}^1 denote consumption, savings in middle age, the number of children, and consumption in old age. They maximize (19) subject to the following budget constraints:

$$w_t^1 A_t = c_t^1 + s_t^1 + z_t n_t^1, \tag{20}$$

$$d_{t+1}^1 = E_t(r_{t+1}) s_t^1 = \begin{cases} r_{t+1}^0 s_t^1 & \text{if state 0 is realized in period } t+1, \\ r_{t+1}^1 s_t^1 & \text{if state 1 is realized in period } t+1. \end{cases} \tag{21}$$

Using Eqs.(19)-(21), the maximization problem can be expressed as

$$\max_{n_t^1, s_t^1} \log(w_t^1 A_t - s_t^1 - z_t n_t^1) + \gamma \log n_t^1 + \beta \left[(1-p_{t+1}) \log r_{t+1}^0 s_t^1 + p_{t+1} \log r_{t+1}^1 s_t^1 \right].$$

The first-order conditions are

$$\frac{-z_t}{w_t^1 A_t - s_t^1 - z_t n_t^1} + \frac{\gamma}{n_t^1} = 0,$$

$$\frac{-1}{w_t^1 A_t - s_t^1 - z_t n_t^1} + \frac{\beta}{s_t^1} = 0.$$

We can derive

$$n_t^1 = \frac{\gamma}{1+\beta+\gamma} \cdot \frac{w_t^1 A_t}{z_t}, \tag{22}$$

$$s_t^1 = \frac{\beta}{1+\beta+\gamma} w_t^1 A_t. \tag{23}$$

The laws of motion for the capital stock and the population are

$$K_{t+1} = s_t^1 N_t, \tag{24}$$

$$N_{t+1} = n_t^1 N_t. \tag{25}$$

Combined Eq.(22) with Eq.(23), these equations imply:

$$K_{t+1}/N_{t+1} = \beta z_t/\gamma = \beta \bar{z} A_t^{\phi}/\gamma. \tag{26}$$

3 Equilibrium

Substituting (18) and (26) into (5) and (7), we obtain:

$$w_t^0 = (1-\alpha)\left(\frac{\beta \bar{z} A_{t-1}^{\phi-1}}{\gamma}\right)^{\alpha} \quad \forall t \geq 2, \tag{27}$$

$$w_t^1 = (1-\alpha)\left(\frac{\beta \bar{z} A_{t-1}^{\phi-1}}{\gamma a}\right)^{\alpha} \quad \forall t \geq 2. \tag{28}$$

Substituting Eqs.(27) and (28) into (14) and (22), the fertility decisions are

$$n_t^0 = \frac{(1-\alpha)\gamma^{1-\alpha}\beta^{\alpha}}{(1+\beta+\gamma)\bar{z}^{1-\alpha}} \frac{1}{A_{t-1}^{(1-\alpha)(\phi-1)}} \qquad \forall t \geq 2, \tag{29}$$

$$n_t^1 = \frac{(1-\alpha)\gamma^{1-\alpha}\beta^{\alpha}}{(1+\beta+\gamma)\bar{z}^{1-\alpha}} \cdot \frac{a^{1-\alpha-\phi}}{A_{t-1}^{(1-\alpha)(\phi-1)}} \qquad \forall t \geq 2. \tag{30}$$

Furthermore, individuals' savings are given by

$$s_t^0 = \frac{\beta}{1+\beta+\gamma}\left(\frac{\beta \bar{z}}{\gamma}\right)^{\alpha} A_{t-1}^{\alpha(\phi-1)+1} \qquad \forall t \geq 2, \tag{31}$$

$$s_t^1 = \frac{\beta}{1+\beta+\gamma}\left(\frac{\beta \bar{z}}{\gamma a}\right)^{\alpha} \cdot a \cdot A_{t-1}^{\alpha(\phi-1)+1} \qquad \forall t \geq 2. \tag{32}$$

To derive (29)–(32), we have used the facts that $A_t = A_{t-1}$ if state 0 is realized and $A_t = aA_{t-1}$ if state 1 is realized in period t. The key steps are as follows. For state 0, substituting w_t^0 from (27) and $z_t = \bar{z} A_{t-1}^{\phi}$ (since $A_t = A_{t-1}$) into (14) yields

$$n_t^0 = \frac{\gamma}{1+\beta+\gamma} \cdot \frac{(1-\alpha)(\beta \bar{z} A_{t-1}^{\phi-1}/\gamma)^{\alpha} \cdot A_{t-1}}{\bar{z} A_{t-1}^{\phi}} = \frac{(1-\alpha)\gamma^{1-\alpha}\beta^{\alpha}}{(1+\beta+\gamma)\bar{z}^{1-\alpha}} \cdot A_{t-1}^{-(1-\alpha)(\phi-1)}.$$

For state 1, substituting w_t^1 from (28) and $z_t = \bar{z}(aA_{t-1})^{\phi}$ (since $A_t = aA_{t-1}$) into (22) gives

$$n_t^1 = \frac{\gamma}{1+\beta+\gamma} \cdot \frac{(1-\alpha)(\beta \bar{z} A_{t-1}^{\phi-1}/(\gamma a))^{\alpha} \cdot aA_{t-1}}{\bar{z}(aA_{t-1})^{\phi}} = \frac{(1-\alpha)\gamma^{1-\alpha}\beta^{\alpha}}{(1+\beta+\gamma)\bar{z}^{1-\alpha}} \cdot \frac{a^{1-\alpha-\phi}}{A_{t-1}^{(1-\alpha)(\phi-1)}}.$$

An important implication of these expressions is that $n_t^1 < n_t^0$ when $\phi > 1 - \alpha$, since $n_t^1 = a^{1-\alpha-\phi} n_t^0$ and $a^{1-\alpha-\phi} < 1$. This result has an intuitive economic interpretation: when technological innovation occurs, the labor-augmenting productivity increases from A_{t-1} to aA_{t-1}, and child-rearing costs increase from $\overline{z}A_{t-1}^{\phi}$ to $\overline{z}(aA_{t-1})^{\phi} = a^{\phi}\overline{z}A_{t-1}^{\phi}$. While wages also increase, the rise in child-rearing costs outpaces the income gain when $\phi > 1$. Specifically, effective income grows by a factor of $a^{1-\alpha}$ (from the wage and technology terms), but child-rearing costs grow by a factor of a^{ϕ}. Since $\phi > 1 > 1 - \alpha$, the cost effect dominates, leading to lower fertility following innovation. Technological progress, while increasing living standards, simultaneously raises the relative cost of child-rearing, thereby suppressing fertility.

For the initial middle-aged agents in period 1, the wage rates are given by

$$w_1^0 = (1-\alpha)\left(\frac{K_1}{A_0 N_1}\right)^{\alpha}, \tag{33}$$

$$w_1^1 = (1-\alpha)\left(\frac{K_1}{aA_0 N_1}\right)^{\alpha}. \tag{34}$$

By substituting Eqs.(33) and (34) into (14) and (22), we can obtain:

$$n_1^0 = \frac{\gamma(1-\alpha)}{(1+\beta+\gamma)\overline{z}}\left(\frac{K_1}{N_1}\right)^{\alpha} A_0^{1-\alpha-\phi}, \tag{35}$$

$$n_1^1 = \frac{\gamma(1-\alpha)}{(1+\beta+\gamma)\overline{z}}\left(\frac{K_1}{N_1}\right)^{\alpha} a^{1-\alpha-\phi} A_0^{1-\alpha-\phi}, \tag{36}$$

while the savings follow

$$s_1^0 = \frac{\beta}{1+\beta+\gamma}\left(\frac{K_1}{N_1}\right)^{\alpha} A_0^{1-\alpha}, \tag{37}$$

$$s_1^1 = \frac{\beta}{1+\beta+\gamma}\left(\frac{K_1}{aN_1}\right)^{\alpha} (aA_0)^{1-\alpha}. \tag{38}$$

An equilibrium path of this economy can be determined as follows.

Definition 1 (Competitive Equilibrium)**.** The equilibrium path is characterized by a sequence of $\{A_{t-1}, K_t, N_t\}_{t=1}^{\infty}$, a sequence of input prices $\{w_t, r_t\}_{t=1}^{\infty}$ and individuals' choices $\{n_t^0, n_t^1, s_t^0, s_t^1\}$ and innovation probability $\{p_t\}_{t=1}^{\infty}$ such that

1. Given A_0, K_1, and N_1, the state of period 1 (with probability p_1) is determined by (8) and N_1. Once the state is determined, A_1 is determined by (2) and A_0. The wage is determined by either (33) or (34), and the interest rate is determined by either (4) or (6), which solves the firm's problem. Given the wage, Eqs.(35)–(38) solve the initial middle-aged agents' problem. Moreover, N_2 is determined using either (17) or (25).

2. For $t \geq 2$ till infinity, given A_{t-1}, N_t, the state of period t (with probability p_t) is determined by (8) and N_t at the beginning of period t. Then, A_t is determined by (2) and A_{t-1}. The wage is determined by either (27) or (28), and the interest rate is determined by either (4) or (6), which solves the firm's problem. Given the wage, Eqs.(29)–(32) solve the middle-aged agents' problem. Moreover, N_{t+1} is determined using either (17) or (25).

One can define $\hat{k}_t \equiv K_t/(A_t N_t)$. From (18), we have $K_{t+1}/N_{t+1} = \beta \bar{z} A_t^{\phi}/\gamma$. In period $t+1$, if state 0 is realized where $A_{t+1} = A_t$, $\hat{k}_{t+1} = \beta \bar{z} A_t^{\phi-1}/\gamma$. If state 1 is realized, ($A_{t+1} = aA_t$), $\hat{k}_{t+1} = \beta \bar{z} A_t^{\phi-1}/(\gamma a)$. Thus, the equilibrium is fully characterized by the evolution of A_t and N_t, with the capital-effective-labor ratio adjusting endogenously through the savings decisions.

4 Long-run population and technology

With the equilibrium specified, we will show how the economy behaves in the long run. Specifically, the economy's population will almost certainly be zero in the long run, and labor productivity will converge stochastically to a finite number of values. The first half of the assertion can be proved as follows.

Lemma 1. *Define $G(\alpha, \beta, \gamma, \bar{z})$ and ω as*

$$G(\alpha, \beta, \gamma, \bar{z}) \equiv \frac{(1-\alpha)\gamma^{1-\alpha}\beta^{\alpha}}{(1+\beta+\gamma)\bar{z}^{1-\alpha}}, \tag{39}$$

$$\omega \equiv \frac{\log G(\alpha, \beta, \gamma, \bar{z}) - (1-\alpha)(\phi-1)\log A_0}{(1-\alpha)(\phi-1)\log a}, \tag{40}$$

and let i^* be the non-negative and smallest integer greater than ω. If $A_t = a^{i^*} A_0$ for some $t (\geq i^*)$, then $\lim_{t\to\infty} N_t = 0$.

Proof. See Appendix A.1. □

Lemma 1 identifies a labor productivity threshold $a^{\omega} A_0$. When labor productivity is below this threshold, the economy experiences positive population growth as the fertility rate is greater than 1. Conversely, the population may decline if labor productivity surpasses this threshold since the fertility rate is below 1. Intuitively, technological progress increases both income and child-rearing costs. When labor productivity is low, the income effect dominates, leading to fertility above replacement. However, since child-rearing costs grow faster than income (due to $\phi > 1$), each innovation step actually reduces fertility relative to the no-innovation outcome. Once labor productivity exceeds the threshold, even the highest possible fertility (which occurs without innovation) falls below 1, and the population begins to decline. When this effect persists, the population experiences constant decay and becomes asymptotically zero in the long run.

Lemma 2. *Let i^* be as defined in Lemma 1. Then*

$$Pr\left(\lim_{t\to+\infty} A_t \geq a^{i^*} A_0\right) = 1, \tag{41}$$

where $Pr(X)$ is the probability that the event X occurs.

Proof. See Appendix A.2. □

Proposition 1. *The population of this economy almost surely converges to zero:*

$$Pr\left(\lim_{t\to+\infty} N_t = 0\right) = 1$$

Proof. Lemma 1 and 2 jointly establish this proposition. □

Lemma 2 implies that labor productivity will almost certainly exceed the threshold established in Lemma 1. For the economy to remain below this threshold, innovations must fail every period as labor productivity approaches this threshold. However, this scenario is highly improbable, as a growing population increases the likelihood of innovation, making it nearly inevitable. Broadly speaking, as the population grows and fuels innovation, technological advancement can and will eventually reach a point where the feedback mechanism diminishes in importance. [6]

5 Numerical Exercise

To see the long-run convergence of labor productivity and the dynamics of other variables, it is useful to consider some numerical simulations.

The capital share is $\alpha = 0.36$, which is commonly used in macroeconomic literature (Galí and Rabanal, 2004). One period in the model is assumed to be 30 years in real life. Thus, the discounting factor is calibrated as $\beta = 0.99^{30}$ (implying an annual interest rate of approximately 1%). The innovation step size is $a = 2$, implying that technology doubles when innovation occurs, which is reasonable given the length of a period in the model. Parameters such as $\bar{z}$ and γ are calibrated to attain positive population growth in the early stage of development. In this example, we set $\bar{z} = 1$ and $\gamma = 0.7\beta$, where the preference for children is proportional to the discount factor. The elasticity of child-rearing cost with respect to labor productivity is set at $\phi = 2$, so that a 1% increase in labor-augmenting productivity is associated with a 2% increase in child-rearing costs[7]. For the initial values, we normalize the population and capital stock size to $N_1 = K_1 = 1$. Finally, the initial labor productivity (technology) is set at a very low level $A_0 = 0.01$, representing an early stage of economic development.

Furthermore, a functional form for the innovation success rate is needed. For the baseline formula, we use $p_t = \psi N_t/(1 + \psi N_t)$, which satisfies all the properties in (9)[8] . The parameter ψ represents the proportion of the working

[6] This concept parallels the notion that the "Industrial Revolution is inevitable," as discussed in Jones (2001). In his paper, population size is a critical factor for innovation. As long as the population is growing, subsequent technological progress is possible. While institutions, such as the establishment of property rights, are essential for economic growth, their primary role is to determine the timing of technological breakthroughs.

[7] According to BLS data, from 2000 to 2024, the costs of day care and preschool in U.S. cities have increased by 153%, or about 6% per year, while real median income has increased by 99%, or about 4% per year.

[8] We emphasize that the qualitative results of the model are robust to alternative functional forms satisfying the same monotonicity and boundary conditions. Alternative formulations, such as $p_t = 1 - e^{-\psi N_t}$ (exponential), $p_t = (\psi N_t)^\lambda/(1 + (\psi N_t)^\lambda)$ (power logistic), or $p_t =$

population (N_t) who are engaged in R&D activities. The denominator $(1 + \psi N_t)$ captures diminishing returns. As the economy grows larger, the additional population has a smaller marginal effect on innovation probability. In the US, the proportion of employed persons who work in R&D (or design) as a primary or secondary occupation ranges from 18% to 22% in 2015 (Burke et al., 2021). For the simulation, we assume a value of $\psi = 0.2$.

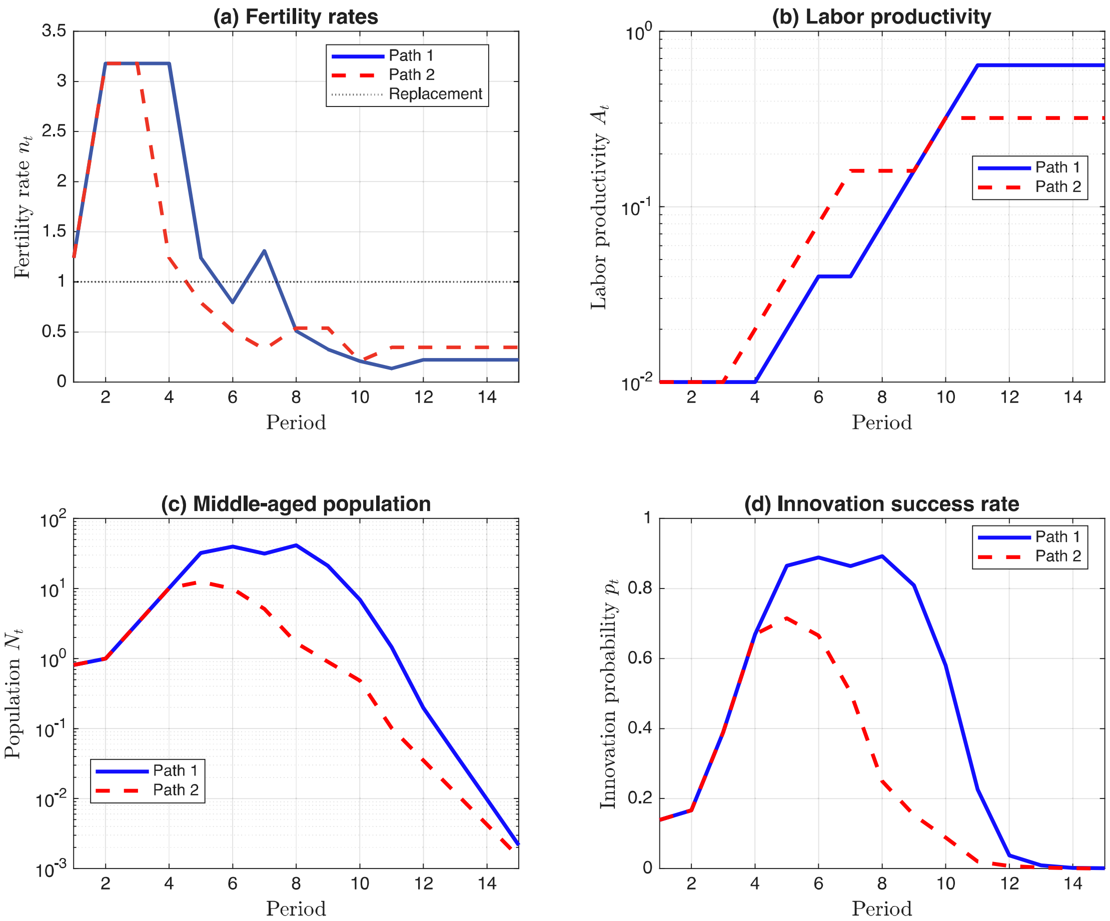


Figure 3: The dynamics of fertility, labor productivity, working middle-aged population, and innovation success rate in two sample paths.

Since growth in the model is stochastic, each iteration of the simulation can generate a different terminal state for technology. First, we show in Figure 3 two sample paths from two simulations to verify Proposition 1 and compare their different development history.

Let us focus on the sample path represented by the solid lines. In this path, the population peaked in period 8 at a level more than 41 times its initial level. Labor productivity is 64 times its initial level. Labor productivity continues to increase until period 11, after which there is no further increase. This is because, by period 11, the population has shrunk to a sufficiently small number, leading to a close-to-zero probability of successful innovation. We can say that

$\ln(1 + \psi N_t)/(1 + \ln(1 + \psi N_t))$ (logarithmic), all satisfy $p'(N) > 0$, $\lim_{N \to 0} p = 0$, $\lim_{N \to \infty} p = 1$ and yield qualitatively similar results, as we demonstrate through numerical simulations in Appendix B.

the long-run state of the economy is realized at this point. This is because what happens in this economy from that period onward is a repetition of the same thing that happens every period, except that the size of the economy shrinks due to population decline. There will be no change in factor prices, fertility decisions, and productivity gain since the population is too small (and so is the talent pool necessary for innovation). As a consequence, every person attains the same level of utility.

For comparison, we simulate another sample path represented by dashed lines to emphasize the potential differences. First, in this case, fertility is lower during the early development phase due to rapid technological advancement. As a result, the size of the economy fails to reach a very high level. It peaks in period 5, with the working population only about 12.5 times its initial level. The lack of population leads to a lower overall innovation success rate and slow improvement in technological progress in the later phase of development. Terminal productivity reaches only 32 times its initial level, compared to 64 times in Path 1.

The previous demonstration shows that long-run labor productivity and welfare can vary significantly, even with identical initial conditions. To explore other potential long-run values, we simulate the economy one million times. The distribution and expected values for key variables are reported in Table 1.

Table 1: Steady State Distributions of Outcomes

Terminal Productivity A^*	Terminal Utility U^*	Distribution (%)
0.16	-6.9843	52.0838
0.32	-5.8231	40.5517
0.64	-4.7409	7.0869
1.28	-3.6588	0.2757
2.56	-2.5767	0.0018
5.12	0.0000	0.0000
Expected value		
0.2588	-6.3460	

Notes: Variables with an asterisk denote terminal (steady-state) values. Distribution shows the percentage of simulation runs reaching each terminal state from 1 million iterations.

These results suggest the economy's long-run behavior follows a steady-state distribution, with extremes in productivity being unlikely. Based on the given parameters, we can calculate from Lemma 1 that $\omega = 3.12$, which implies that after the 4th innovation step is achieved, fertility will reach the subreplacement level. Since this economy is destined to attain this level, the terminal state of labor productivity is never below $A_0 \times 2^4 = 0.16$. In all states, the terminal population is asymptotically zero. Meanwhile, the expected long-run value of labor productivity is 0.2588, and welfare is -6.3460.

Note that the concept we used for welfare comparison is of Millian efficiency (Mill, 1884) instead of Benthamite criterion (Bentham, 1879). In the former, welfare is valued by per-capita utility, whereas in the latter, it is valued by the sum of population utilities. In our model, since the population size is endogenous, using the latter criterion makes it difficult to analyze welfare as we need to evaluate the welfare of the unborn (Golosov et al., 2007). Therefore, for the remainder of the analysis, we follow Conde-Ruiz et al. (2010) and use only the information of the alive agents and their preference profiles (equations (11) and (19)) when examining welfare outcomes.

6 Effects of a pronatalist policy

The previous section shows that economic welfare differs greatly depending on the long-term state to be reached. We wonder if it is possible to attain better living standards, i.e., increase the likelihood of achieving them, in the long run.

In this model, the decision of an economic agent to have a child has an external impact on the likelihood of successful innovation in the future. Suppose an economic agent chooses to have another child now. In that case, it raises the chances of innovation by increasing the population in the next period, which in turn stochastically increases the income and child-rearing costs for the next middle-aged generation. In the early stages of economic growth, it may be justifiable to encourage childbearing through policy because the positive external effects of higher incomes still outweigh the negative effects of increased child-rearing costs. In other words, with a pronatalist policy, the economy can experience as much innovation and labor productivity growth as possible during the early stages of development, which, in turn, benefits future generations.

6.1 Child-rearing support

We now consider a pronatalist policy that aims to alleviate some of the costs associated with child-rearing. The government gives a subsidy amount of $b(>0)$ per child to all middle-aged parents. Once enacted, the subsidy is maintained for all current and subsequent generations. To finance this policy, the government imposes a lump-sum tax of τ_t on the middle-aged agents' income in period t.

To distinguish this case from the case without subsidy, we denote the economic decisions when subsidy exists by the superscript s. The real cost of child-rearing is now given by

$$z_t^s = \bar{z} A_t^{\phi} - b. \tag{42}$$

For the realized state $i \in \{0, 1\}$, a middle-aged agent's budget constraints are:

$$w_t^{i,s} A_t^i - \tau_t^i = c_t^{i,s} + s_t^{i,s} + z_t^s n_t^{i,s}, \tag{43}$$

$$d_{t+1}^{i,s} = \begin{cases} r_{t+1}^0 s_t^{i,s} & \text{if state 0 is realized in period } t+1, \\ r_{t+1}^1 s_t^{i,s} & \text{if state 1 is realized in period } t+1 \end{cases} \tag{44}$$

A household's problem can now be formulated as

$$\max_{s_t^{i,s},n_t^{i,s}} \log(w_t^{i,s}A_t^i - \tau_t^i - s_t^{i,s} - z_t^s n_t^{i,s}) + \gamma \log n_t^{i,s}$$
$$+ \beta E_t[(1-p_{t+1})\log r_{t+1}^0 s_t^{i,s} + p_{t+1}\log r_{t+1}^1 s_t^{i,s}].$$

The first-order conditions are

$$\frac{z_t^s}{w_t^{i,s}A_t^i - \tau_t^i - s_t^{i,s} - z_t^s n_t^{i,s}} = \frac{\gamma}{n_t^{i,s}}, \tag{45}$$

$$\frac{1}{w_t^{i,s}A_t^i - \tau_t^i - s_t^{i,s} - z_t^s n_t^{i,s}} = \frac{\beta}{s_t^{i,s}}. \tag{46}$$

Solving the system yields

$$n_t^{i,s} = \frac{\gamma}{1+\beta+\gamma}\frac{w_t^{i,s}A_t^i - \tau_t^i}{z_t^s}, \tag{47}$$

$$s_t^{i,s} = \frac{\beta}{1+\beta+\gamma}(w_t^{i,s}A_t^i - \tau_t^i). \tag{48}$$

Furthermore, the government's budget is balanced every period such that:

$$\tau_t^i = bn_t^i. \tag{49}$$

When state 0 is realized, $A_t = A_{t-1}$. Eqs. (47) and (48) become

$$n_t^{0,s} = \frac{\gamma}{1+\beta+\gamma}\frac{w_t^{0,s}A_{t-1} - \tau_t^0}{z_t^s}, \tag{50}$$

$$s_t^{0,s} = \frac{\beta}{1+\beta+\gamma}(w_t^{0,s}A_{t-1} - \tau_t^0). \tag{51}$$

The law of motion for the capital per person reads:

$$\frac{K_{t+1}}{N_{t+1}} = \frac{s_t^{0,s}N_t}{n_t^{0,s}N_t} = \beta z_t^s/\gamma = \beta(\bar{z}A_t^{\phi} - b)/\gamma. \tag{52}$$

Inserting (52) into (5), we obtain the following wage rate

$$w_t^{0,s} = (1-\alpha)\left(\frac{K_t}{A_{t-1}N_t}\right)^{\alpha} = (1-\alpha)\left(\frac{\beta}{\gamma}\right)^{\alpha}\left(\frac{\bar{z}A_{t-1}^{\phi} - b}{A_{t-1}}\right)^{\alpha}. \tag{53}$$

Eqs.(49),(50),(53) jointly imply

$$n_t^{0,s} = \frac{\gamma}{1+\beta+\gamma}\left[\frac{(1-\alpha)(\beta/\gamma)^{\alpha}(\bar{z}A_{t-1}^{\phi} - b)^{\alpha}A_{t-1}^{1-\alpha} - bn_t^{0,s}}{\bar{z}A_{t-1}^{\phi} - b}\right].$$

Collecting $n_t^{0,s}$ terms yields

$$n_t^{0,s} = \frac{(1-\alpha)\gamma^{1-\alpha}\beta^{\alpha}}{1+\beta+\gamma} \cdot \frac{\left(\dfrac{A_{t-1}}{\bar{z}A_{t-1}^{\phi}-b}\right)^{1-\alpha}}{1+\dfrac{\gamma}{1+\beta+\gamma}\cdot\dfrac{b}{\bar{z}A_{t-1}^{\phi}-b}}. \tag{54}$$

Similarly, we can also derive the fertility of when state 1 realized:

$$n_t^{1,s} = \frac{(1-\alpha)\gamma^{1-\alpha}\beta^{\alpha}}{1+\beta+\gamma} \cdot \frac{\left(\dfrac{aA_{t-1}}{\bar{z}(aA_{t-1})^{\phi}-b}\right)^{1-\alpha}}{1+\dfrac{\gamma}{1+\beta+\gamma}\cdot\dfrac{b}{\bar{z}(aA_{t-1})^{\phi}-b}}. \tag{55}$$

It is easy to verify that when $b=0$, (54) is equivalent to (29) and (55) is equivalent to (30). We are now interested in evaluating the effectiveness of this policy.

First, we should note that determining the size of the subsidy b also determines the timing of the policy's introduction. As A_t cannot take any values other than $A_0, aA_0, a^2A_0, a^3A_0, \ldots$. Given a value of b, there must be an integer i^* such that

$$\bar{z}(a^{i^*-1}A_0)^{\phi} < b < \bar{z}(a^{i^*}A_0)^{\phi},$$

where the value of b is chosen such that $b < \bar{z}(a^iA_0)^{\phi}$ for any non-negative integer i. Let t^* be the first period when the following equation is established

$$A_{t^*} = a^{i^*}A_0.$$

Then, the subsidy policy will not be implemented until period t^*-1 and remain enacted in and after period t^*. As a result, when the policy is implemented this way, the optimization behavior differs between the middle-aged generation up to period t^*-1 and the middle-aged generation after period t^*. Specifically, the former will not benefit from the subsidy, while the latter will benefit from it.

Second, we also need to check if the purpose of lifting the fertility rate is fulfilled. In other words, given a chosen policy b, we are interested in finding the timing t^* such that the fertility decisions made by all generations in and after t^* are improved. Comparing the fertility decisions (54) with (29), and (55) with (30), we obtain the following result:

Proposition 2. *Let A_q be a value satisfying*

$$(\bar{z}A_q^{\phi}-b)^{\alpha}[(\bar{z}A_q^{\phi})^{1-\alpha}-(\bar{z}A_q^{\phi}-b)^{1-\alpha}] - \frac{\gamma}{1+\beta+\gamma}b = 0.$$

then we have:

$$n_t^{0,s} > n_t^0 \text{ and } n_t^{1,s} > n_t^1 \;\forall t \geq t^* \text{ where } A_{t^*} > A_q.$$

Given a subsidy level b first introduced in t^* satisfying the above, then the fertility is always higher than the case without policy, regardless of the state realized in the generations after that.

Proof. See Appendix A.3 □

Proposition 2 implies that once the subsidy amount b is decided, the government should implement the policy as soon as the productivity level gets sufficiently large. By maintaining the same subsidy level from the time it is first introduced, the fertility rates of all subsequent generations will be improved, regardless of which state realizes.

In terms of welfare, since this policy attempts to encourage middle-aged agents to spend their income on child-rearing rather than consumption and savings, a reduction in the current generation's lifetime utility is expected. However, the policy can potentially increase future generations' expected utility in the long run by increasing the likelihood that higher productivity can be realized. This stochastic nature of the model suggests that the reduction in welfare of some generations after the policy's implementation is only temporary.

To examine this point, let us consider the case when the government wants to implement a child-rearing subsidy of $b = 0.02$ under the previous simulations' parameters. The threshold $A_q = 0.1484$ implies that the subsidy is implemented at and after period t^* where $A_{t^*} = 0.16$. We iterate the simulations one million times and report the steady-state distribution in Table 2.

Table 2: Steady State Distributions of Outcomes with Subsidy Policy

Terminal Productivity	Terminal Utility	Distribution (%)	
A^*	U^*	$b = 0.02$	$b = 0$
0.16	−6.9843	16.80	52.08
0.32	−5.8231	51.65	40.55
0.64	−4.7409	26.93	7.09
1.28	−3.6588	4.42	0.28
2.56	−2.5767	0.20	0.00
5.12	0.0000	0.00	0.00
Expected value			
0.4232	−5.8409	(With policy $b = 0.02$)	
0.2590	−6.3452	(Without policy $b = 0$)	

Notes: Variables with an asterisk denote terminal (steady-state) values. The policy consists of a child subsidy of $b = 0.02$. Distribution shows the percentage of simulation runs reaching each terminal state.

Compared to Table 1 where no policy is implemented, the subsidy has reduced the probabilities of states with low long-run productivity levels and increased the probabilities of states with high long-run productivity levels. Although the population, in the long run, is still asymptotically zero, the policy has improved the expected long-run labor productivity and welfare of future generations by 63.39% and 8.56%, respectively.

6.2 Welfare analysis

Table 2 only gives information about the welfare of the distant generations (when innovations have already been exhausted). Indeed, it is also important to pay attention to the potential loss in welfare during the transition following the implementation of the policy. Specifically, the welfare of the current generation may be reduced as they (have to) cut consumption in favor of more childbearing. This action is meant to change the population size of the next generation. However, whether such a change happens is stochastic due to the nature of innovation. Furthermore, the timing of policy implementation can be different in each iteration of the simulation, which changes the timing of the welfare deficit. Overall, the stochastic nature of growth makes it difficult to directly compare the welfare before and after the policy's introduction.

To properly compare the expected welfare for the generations with and without policy intervention, we rely on the intergenerational average welfare. The computation proceeds as follows. First, following the principle of Millian efficiency (Mill, 1884; Conde-Ruiz et al., 2010), we only consider the per-capita utility of agents who are alive in each generation, abstracting from the welfare of the unborn. Then, we simulate the economy for 100 generations and iterate this process one million times. This yields one million independent sample trajectories, each representing a possible realization of the economy's stochastic path from generation 0 to generation 100.[9] Then, for each generation t, we compute its expected utility by averaging the realized utility values across all one million trajectories. This gives us the expected welfare that a representative agent in generation t would attain.

Finally, to compare dynasties across the two policy scenarios (with and without child-rearing subsidy), we sequentially aggregate the expected welfare of successive generations and then compute the average value. For instance, to compute the average intergenerational welfare up to the generation j, we add the expected welfare of generation 1, that of generation 2, and so on until generation j, and then divide by the number of generations, all without discounting future generations' welfare. It is worth pointing out that this measure treats all generations equally, as it gives equal weight to each generation's well-being without any discounting, similar to a typical Social planner's problem. Furthermore, the purpose of averaging intergenerational welfare is to account for the potential reduction in future generations' welfare that may result from introducing the pronatalist policies.

We calculate this measurement in two scenarios, one without policy ($b = 0$) and with a pronatalist policy implemented ($b = 0.02$). This value gives us information about the short-run (low j) and long-run effects (high j) of the pronatalist policy. A high j means that we are counting more distant generations that may benefit from the policy. Vice versa, a low j implies we are counting more old generations, who are likely to suffer immediate welfare loss following the implementation of the policy. Figure 4 shows this tradeoff.

In the short run, the pronatalist policy can lead to some welfare losses. As

[9] Although each genealogy spans to infinity, the information generated for 100 periods is sufficient for the comparison.

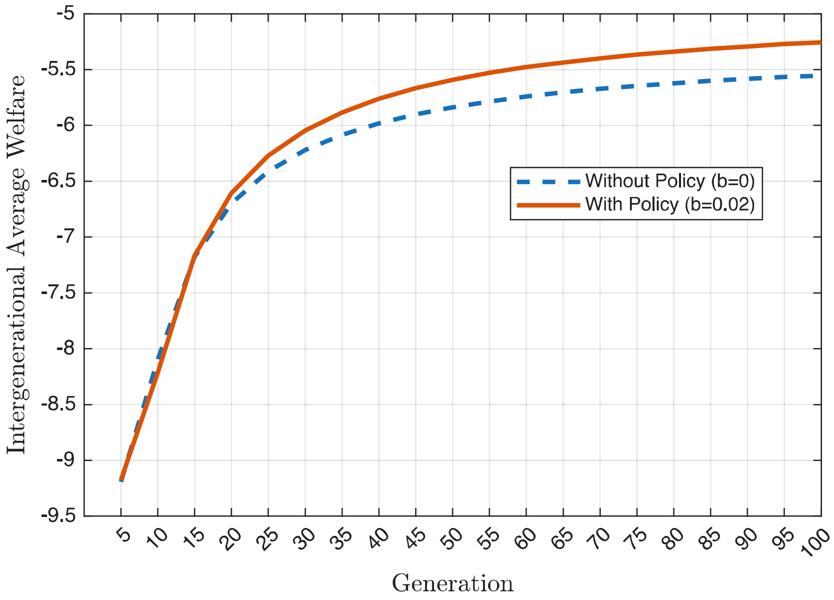


Figure 4: Inter-generational average welfare with and without policy.

shown in Figure 4, the inter-generational average welfare until generation 30 is worse off than the case without policy. This is simply because individuals are taxed to consume less in order to give more birth. However, by effectively improving the likelihood of having more children, the expected welfare of future generations is improved because the economy experiences more innovation steps. Ultimately, the policy is effective at improving the average welfare of the economy in the long run. In other words, as time goes on, the benefits to future generations will outweigh the temporary costs incurred to the present.

In what follows, we examine the effects of different levels of child-rearing subsidies. Should the government implement a small subsidy in the early stage of development or a bigger subsidy at a later stage? Figure 5 presents the long-run expected values of labor productivity and welfare (averaging one million iterations) for each subsidy level ranging from 0.0003 to 0.08. The smaller subsidy value implies earlier policy implementation, while the bigger subsidy requires higher labor productivity in the later stage of development in order to be implemented.

We find that any level of child-rearing subsidy can enhance long-run labor productivity by encouraging childbearing, which in turn increases population size and the rate of innovation throughout the economy's development. Even a small subsidy introduced early can have substantial and lasting effects, as it incrementally improves the fertility rates of all subsequent generations. While the positive impact of the policy on productivity is evident, what happens to welfare is rather mixed. If the subsidy is too large, the resulting need for higher taxes on parental income may reduce long-run welfare. Additionally, given the stochastic nature of growth, there is a risk that the economy may have already exhausted its population resources needed for innovation before the subsidy can make a significant impact. The analysis of optimal timing for a policy is difficult to conduct since growth deployed in this model is stochastic by nature. Nevertheless, it is safe to say that implementing a child-rearing subsidy policy

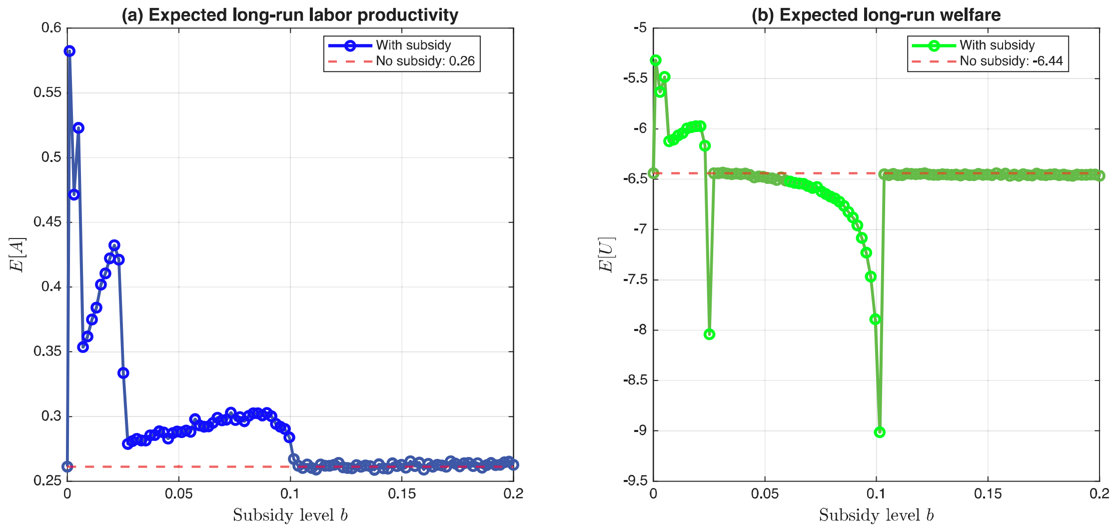


Figure 5: Detailed simulations for a wide range of child-rearing subsidy level b.

at an earlier stage has better outcomes than later implementation.

7 Conclusion

This paper develops an overlapping-generations model with endogenous fertility and stochastic labor-augmenting technological progress to examine the long-run dynamics of labor productivity and population, particularly in the context of potential global population decline. The key assumption is the convex child-rearing cost function with respect to labor productivity. In the early stage of development, when the income effect dominates fertility decisions, an economy can experience rapid growth in both population and labor productivity. Once labor productivity reaches a sufficiently high level, the increasing child-rearing costs dominate, resulting in fertility decline below the replacement level. The economy then faces a sharp population contraction, reducing innovation power. Eventually, the welfare level stagnates at some level while the population collapses in the long run.

We explore the effects of a pronatalist policy that provides child-rearing subsidies to middle-aged agents. This policy can boost long-run labor productivity at any subsidy level since childbearing is encouraged. However, the effects on welfare reveal crucial results on the timing of the policy. Late implementations imply that the subsidy amount must be sufficiently large to boost fertility rates, which necessitates higher taxes and may fail to be effective. On the other hand, early implementations are more beneficial since they require a smaller tax size, and the fertility rates of more generations can benefit from it.

Several limitations of the model should be acknowledged. First, the assumption that child-rearing costs are convex in technology, while supported by empirical evidence on childcare costs outpacing wage growth, is a simplification. In practice, child-rearing costs encompass both time costs (opportunity costs of parental labor) and financial costs, and their relationship with technology requires a microfoundation framework. Second, the assumption of no technological obsolescence, while plausible for frontier economies, may not hold for developing countries experiencing technological catch-up or regression. Our model is primarily intended to describe long-run dynamics at the technological frontier, where the persistence of knowledge is a reasonable assumption. For countries like Japan, where TFP growth has stagnated, the model can be interpreted as predicting that once the economy reaches sufficiently high productivity, further innovation becomes unlikely due to the declining population, which is broadly consistent with Japan's experience of prolonged stagnation accompanied by population decline. Third, the specific functional form for innovation probability is chosen for tractability. In order to obtain a more micro-founded behavioral equation, a full-fledged microfoundation framework with a dedicated R&D sector is needed. Likewise, the utility function can be modified, as in Tran and Kitagawa (2025) and Baudin et al. (2019), by setting a lower bound on fertility. In such a way, the extreme case of an asymptotic zero population can be avoided. These avenues are left for future research.

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A Proofs

A.1 Proof of Lemma 1

Proof. From (39),(40), we can easily verify that

$$\frac{(1-\alpha)\beta^{\alpha}\gamma^{1-\alpha}}{(1+\beta+\gamma)\overline{z}^{1-\alpha}}(a^{\omega}A_0)^{-(1-\alpha)(\phi-1)} = 1. \tag{56}$$

From the fertility equations (29) and (30), we have $n_t^1 = a^{1-\alpha-\phi}n_t^0 < n_t^0$ since $a > 1$ and $\phi > 1-\alpha$. Since $i^* > \omega$, Eq.(56) implies:

$$\underbrace{\frac{(1-\alpha)\beta^{\alpha}\gamma^{1-\alpha}}{(1+\beta+\gamma)\overline{z}^{1-\alpha}}(\overbrace{a^{i^*}A_0}^{A_{t-1}})^{-(1-\alpha)(\phi-1)}}_{n_t^0 \text{ when } A_{t-1}=a^{i^*}A_0} < 1. \tag{57}$$

Note that the last inequality follows from $i^* > \omega$ and the definition of ω. Since $n_t^1 < n_t^0$ and $A_s \geq A_t$ for any $s > t$, these inequalities mean that

$$\forall s > t, \qquad n_s^0 \leq n_t^0 < 1,$$
$$\forall s > t, \qquad n_s^1 \leq n_t^1 < n_t^0 < 1,$$

so that

$$\forall s > t, \qquad 0 < N_s \leq (n_t^0)^{s-t}N_t. \tag{58}$$

Since $\lim_{s\to+\infty}(n_t^0)^{s-t}N_t = 0$, the inequalities (58) imply that $\lim_{s\to+\infty} N_s = 0$, which is our desired result. □

A.2 Proof of Lemma 2

Proof. Let ω be as defined in Lemma 1. When $\omega < 0$, the value of i^* is determined as $i^* = 0$ and thus (41) is reduced to

$$Pr(\lim_{t\to+\infty} A_t \geq A_0) = 1,$$

which is obviously true.

When $\omega \geq 0$, there exist some non-negative integers that are no larger than ω. Let j be such an integer and consider the probability of the next event occurring:

$$\lim_{t\to+\infty} A_t = a^j A_0.$$

If the j-th innovation occurred in period s, then no innovation occurs after that, so we obtain the following equation:

$$Pr(\lim_{t\to+\infty} A_t = a^j A_0) = \sum_{s=j}^{\infty} P_s \prod_{k=1}^{\infty}(1-p_{s+k}),$$

where P_s is the probability that the j-th innovation occurred in the period s. In addition, the population growth rate after period s is given by

$$\forall k \geq 1,\ n^0_{s+k} = \frac{(1-\alpha)\beta^\alpha \gamma^{1-\alpha}}{(1+\beta+\gamma)\overline{z}^{1-\alpha}} (a^j A_0)^{-(1-\alpha)(\phi-1)} \geq 1.$$

The inequality is obtained from the fact that $j \leq \omega$ and $n^0_t > n^1_t$ for all t. By the definition of p in (8), this inequality implies

$$\forall k \geq 2,\ 1 > \underbrace{1 - p_{s+1}}_{\frac{1}{1+\psi N_{s+1}}} \geq \underbrace{1 - p_{s+k}}_{\frac{1}{1+\psi N_{s+k}}},$$

and thus that

$$\prod_{k=1}^{T} (1 - p_{s+k}) \leq (1 - p_{s+1})^T.$$

From the last inequality, we obtain

$$0 \leq \prod_{k=1}^{\infty} (1 - p_{s+k}) \leq \lim_{T \to +\infty} (1 - p_{s+1})^T = 0.$$

Hence, we can state that

$$\prod_{k=1}^{\infty} (1 - p_{s+k}) = 0,$$

which implies

$$Pr(\lim_{t \to +\infty} A_t = a^j A_0) = 0.$$

The last equation holds true for any integer $j \in [0, i^* - 1]$, meaning that (41) is true again for this case. □

A.3 Proof of Proposition 2

Proof. Comparing (54) with (29), showing

$$n^{0,s}_t > n^0_t$$

is equivalent to verifying

$$\frac{\left(\dfrac{A_{t-1}}{\overline{z} A^\phi_{t-1} - b}\right)^{1-\alpha}}{1 + \dfrac{\gamma}{1+\beta+\gamma} \cdot \dfrac{b}{\overline{z} A^\phi_{t-1} - b}} > \frac{1}{\overline{z}^{1-\alpha} A^{(1-\alpha)(\phi-1)}_{t-1}}.$$

With all terms positive, this reduces to:

$$(\bar{z}A_{t-1}^{\phi}-b)^{\alpha}[(\bar{z}A_{t-1}^{\phi})^{1-\alpha}-(\bar{z}A_{t-1}^{\phi}-b)^{1-\alpha}]>\frac{\gamma}{1+\beta+\gamma}b. \quad (59)$$

Suppressing the time notation, define a new variable $y>b$ such that $y=\bar{z}A_{t-1}^{\phi}$. The inequality (59) simplifies to:

$$f(y)=(y-b)^{\alpha}(y^{1-\alpha}-(y-b)^{1-\alpha})-\frac{\gamma}{1+\beta+\gamma}b>0.$$

Differentiating $f(y)$, we obtain

$$f'(y)=\alpha\left(\frac{y}{y-b}\right)^{1-\alpha}+(1-\alpha)\left(\frac{y}{y-b}\right)^{-\alpha}-1. \quad (60)$$

In what follows, we show that the sign of (60) is positive $\forall y>b$. Define a new variable $\xi=\dfrac{y-b}{y}$. It is clear that $0<\xi<1$ and $\xi'(y)>0$. Eq. (60) becomes:

$$f'(y)=\Psi(\xi),$$

where $\Psi(\xi)=\alpha\xi^{\alpha-1}+(1-\alpha)\xi^{\alpha}-1$. For $\xi\in(0,1)$, we can verify that:

$$\Psi'(\xi)<0, \lim_{\xi\to 1}\Psi(\xi)=0,\ \lim_{\xi\to 0^+}\Psi(\xi)=+\infty,$$

implying that $\Psi(\xi)$ is a strictly decreasing function that tends from $+\infty$ to 0 as ξ tends from 0 to 1. It follows that $\Psi(\xi)>0$ for any $\xi\in(0,1)$. Thus, $f'(y)>0$ for all $y>0$, implying $f(y)$ is strictly increasing.

For any $b>0$, we have:

$$\lim_{y\to b^+}f(y)=-\frac{\gamma}{1+\beta+\gamma}b<0, \qquad \lim_{y\to+\infty}f(y)=+\infty>0.$$

By the Intermediate Value Theorem, there exists a $\tilde{y}\in(b,+\infty)>0$ such that $f(\tilde{y})=0$. Define a variable A_q such that

$$A_q=(\tilde{y}/\bar{z})^{1/\phi}. \quad (61)$$

Since $f'(y)>0$, $f(\tilde{y})=0$, the definition of y and (61) imply

$$n_t^{0,s}>n_t^0\ \forall t\geq t^*\text{ if }A_{t^*}>A_q.$$

Similarly, comparing (55) with (30), we obtain

$$n_t^{1,s}>n_t^1\ \forall t\geq t^{**}\text{ if }A_{t^{**}}>A_{q'}.$$

where $A_{q'}=(\tilde{y}/\bar{z})^{1/\phi}/a$. Since $a>1$, $A_q>A_{q'}$, we can conclude that

$$n_t^{0,s}>n_t^0\text{ and }n_t^{1,s}>n_t^1\ \forall t\geq t^*\text{ if }A_{t^*}>A_q.$$

□

B Alternative Innovation Probability Functions

In this appendix, we demonstrate that the main results of the paper are robust to alternative specifications of the innovation probability function $p(N)$. Recall that the key requirements are:

1. $p'(N) > 0$ (innovation probability is increasing in population),
2. $\lim_{N \to 0} p(N) = 0$ (no innovation with zero population),
3. $\lim_{N \to \infty} p(N) = 1$ (innovation is certain with large population).

We consider the following three alternative functional forms, all of which satisfy conditions 1–3:

1. **Exponential:** $p_t = 1 - e^{-\psi N_t}$. This form arises naturally from a Poisson process where each individual independently generates innovation breakthroughs at a rate ψ.
2. **Power logistic:** $p_t = (\psi N_t)^{\lambda} / (1 + (\psi N_t)^{\lambda})$ with $\lambda = 0.5$. This generalization introduces stronger diminishing returns to population in the innovation process.
3. **Logarithmic:** $p_t = \ln(1 + \psi N_t) / (1 + \ln(1 + \psi N_t))$. This form features the slowest convergence to 1, capturing extreme diminishing returns.

Using the same baseline parameters ($\alpha = 0.36$, $\beta = 0.99^{30}$, $\gamma = 0.7\beta$, $\bar{z} = 1$, $\phi = 2$, $a = 2$, $\psi = 0.20$, $A_0 = 0.01$, $N_1 = 1$), we simulate the economy one million times under each specification and report the distribution results.

The proofs of Proposition 1 require only that $p(N)$ is continuous, increasing in N, and bounded between 0 and 1. All four specifications satisfy these conditions, and thus Proposition 1 (almost sure convergence to zero population) holds for each of them. The key step in the proof of Lemma 2, showing that $\prod_{k=1}^{\infty}(1 - p_{s+k}) = 0$ when the population is non-declining, requires only that p_{s+k} is bounded away from zero, which holds for all four forms when $N > 0$.

Table 3 reports the steady-state distribution of long-run labor productivity A^* under the four specifications. The key takeaway is that all four functional forms produce the same qualitative patterns. Similar results hold true when we experiment with the same pronatalist policy with child subsidies, as can be seen in Table 4. We also perform an intergenerational welfare comparison similar to Figure. 4 under different specifications of $p(N)$. The results from this exercise are shown in Figure 6. One can see that in the short run, the welfare of the first few generations since the introduction of the policy suffers utility decline as a portion of wealth must now be spent on child-rearing instead of consumption. However, since the pronatalist policy increases the likelihood of successful technological progress, the improvement will gradually emerge in the long run.

Table 3: Steady-State Distribution of A^* Under Alternative $p(N)$ Specifications

Terminal Productivity	Terminal Utility	Distribution (%)			
A^*	U^*	Baseline	Exponential	Power	Log
0.16	−6.9841	52.07	60.89	22.07	44.54
0.32	−5.8231	40.57	35.55	48.07	42.23
0.64	−4.7409	7.09	3.50	25.08	11.99
1.28	−3.6588	0.26	0.06	4.50	1.19
2.56	−2.5767	0.00	0.00	0.28	0.04
5.12	0.0000	0.00	0.00	0.01	0.00
10.24	0.0000	0.00	0.00	0.00	0.00
Expected values					
productivity $\mathbb{E}[A^*]$		0.2589	0.2307	0.4146	0.2969
welfare $\mathbb{E}[U^*]$		−6.3452	−6.4927	−5.6862	−6.1830

Notes: Variables with an asterisk denote terminal (steady-state) values. Distribution shows the percentage of simulation runs reaching each terminal state based on one million simulations. Baseline, Exponential, Power, and Logarithmic refer to alternative functional form specifications for the innovation probability $p(N)$.

Table 4: Policy fffectiveness across alternative $p(N)$ specifications

	No Policy ($b = 0$)		With Policy ($b = 0.02$)		Improvement	
Form	$\mathbb{E}[A^*]$	$\mathbb{E}[U^*]$	$\mathbb{E}[A^*]$	$\mathbb{E}[U^*]$	A^* (%)	U^*
Baseline	0.2589	−6.3453	0.4237	−5.8385	63.63	0.5068
Exponential	0.2306	−6.4935	0.3730	−6.0351	61.74	0.4584
Power (0.5)	0.4145	−5.6863	0.6690	−5.0400	61.42	0.6462
Logarithmic	0.2967	−6.1837	0.4875	−5.6492	64.33	0.5345

Notes: Results based on one million simulations. The policy consists of a child subsidy at rate $b = 0.02$. Improvement in A^* is calculated as the percentage increase in expected terminal productivity. Improvement in U^* is the absolute increase in expected terminal utility.

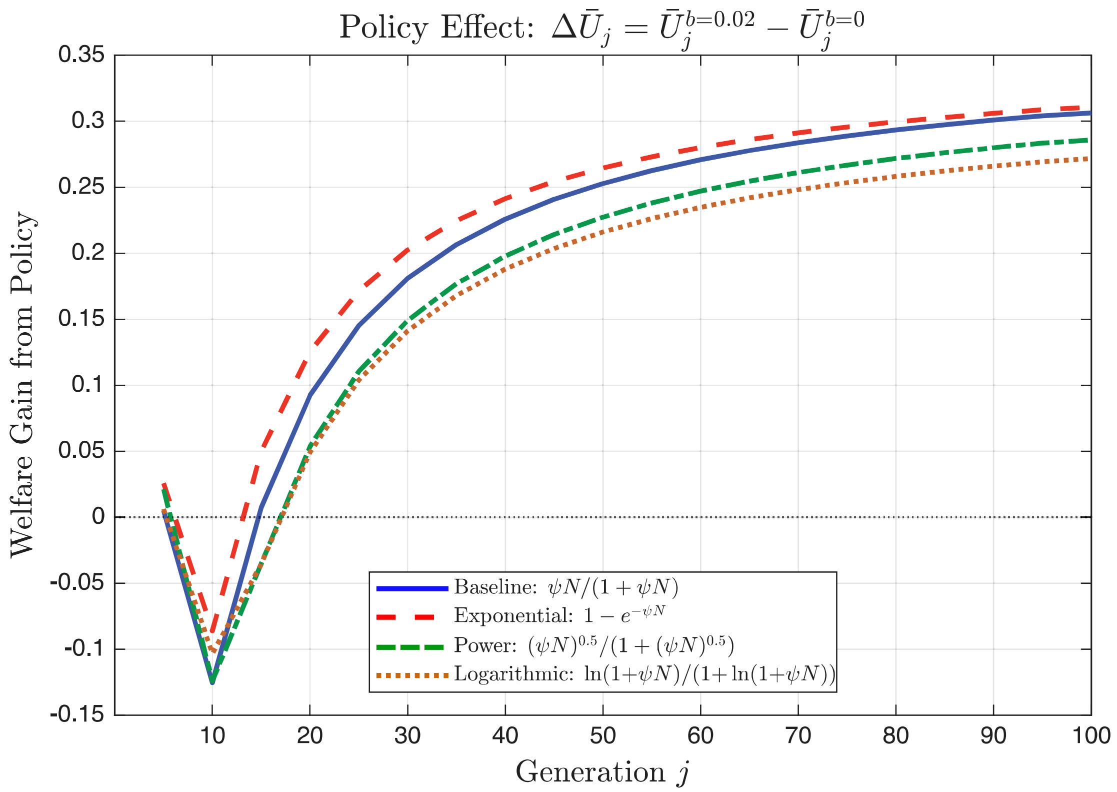

Figure 6: Inter-generational average welfare difference between the case with and the case without policy under different functional forms

C Alternative Child-Rearing Cost Formulation

In this appendix, we consider an alternative formulation of child-rearing costs where costs are proportional to both technology and the wage rate:

$$z_t = \overline{z} A_t^{\phi} w_t. \tag{62}$$

Consider the household's problem when state 0 is realized. The budget constraint becomes:

$$c_t^0 + s_t^0 + z_t n_t^0 = w_t^0 A_t,$$

where $z_t = \overline{z} A_t^{\phi} w_t^0$ under the alternative formulation.

The first-order condition for fertility yields:

$$n_t^0 = \frac{\gamma}{1+\beta+\gamma} \cdot \frac{w_t^0 A_t}{z_t} = \frac{\gamma}{1+\beta+\gamma} \cdot \frac{w_t^0 A_t}{\overline{z} A_t^{\phi} w_t^0} = \frac{\gamma}{(1+\beta+\gamma)\overline{z}} \cdot A_t^{1-\phi}.$$

Similarly, when state 1 is realized with $A_t = aA_{t-1}$:

$$n_t^1 = \frac{\gamma}{1+\beta+\gamma} \cdot \frac{w_t^1 A_t}{\overline{z} A_t^{\phi} w_t^1} = \frac{\gamma}{(1+\beta+\gamma)\overline{z}} \cdot A_t^{1-\phi} = \frac{\gamma}{(1+\beta+\gamma)\overline{z}} \cdot (aA_{t-1})^{1-\phi}.$$

Notice that in both expressions above, the wage rate per unit of efficiency labor w_t cancels out entirely from the fertility decision, and fertility depends only on the technology level. However, because there is no technological obsolescence, fertility rises with productivity and remains high even when there is no technological progress, eliminating the mechanism that would reduce it. On the other hand, if we assume $\phi = 1$, then fertility becomes a constant, and so no dynamical analysis of fertility can be done.