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Dual Caregiving, Declining Birth Rate, and Economic Sustainability *

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Abstract

This paper employs an overlapping generations model to analyze how placing the burden of caring for both elderly parents and children on the working generation shapes fertility and other economic outcomes. In the model, fertility decisions create intergenerational spillovers. When one generation has fewer children, the next generation faces a heavier caregiving burden for its elderly parents, which in turn discourages childbearing. The model reveals sharply different long-run trajectories depending on the time intensity of caregiving. If care demands are moderate, sustainable growth remains feasible despite these externalities. However, when care becomes highly time-intensive, fertility declines, labor supply contracts, and the economy risks falling into a "nursing hell," where most time is devoted to caregiving. Policy measures, such as child allowances, can alleviate this dynamic by expanding the number of siblings and reducing the per-capita caregiving burden. Yet if care demands are extremely high from the outset, even such interventions cannot avert structural collapse.

Keywords: dual caregiving, endogenous fertility, overlapping generations, sustainability

JEL: E13, J13, J14, J22, J24, O11

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1 Introduction

Over the past few decades, many countries have witnessed remarkable increases in life expectancy. Figure 1 shows that by 2021, average life expectancy had surpassed 80 years in many regions. Yet longer lives also pose new challenges. A major concern is that many older individuals spend more than a decade at the end of their lives, unable to live independently, a period comparable to one-third of a typical working life. At the same time, child-rearing has become increasingly time-intensive (Doepke et al., 2023). Together, these trends have created the problem of "dual caregiving," in which middle-aged adults must care simultaneously for children and aging parents. Estimates suggest that these adults constitute about 22.9% in Japan (Yamashita and Soma, 2025), 27% in Europe (Birtha and Holm, 2017), and 23% in the United States (Parker and Patten, 2013), with women comprising the majority.[1]

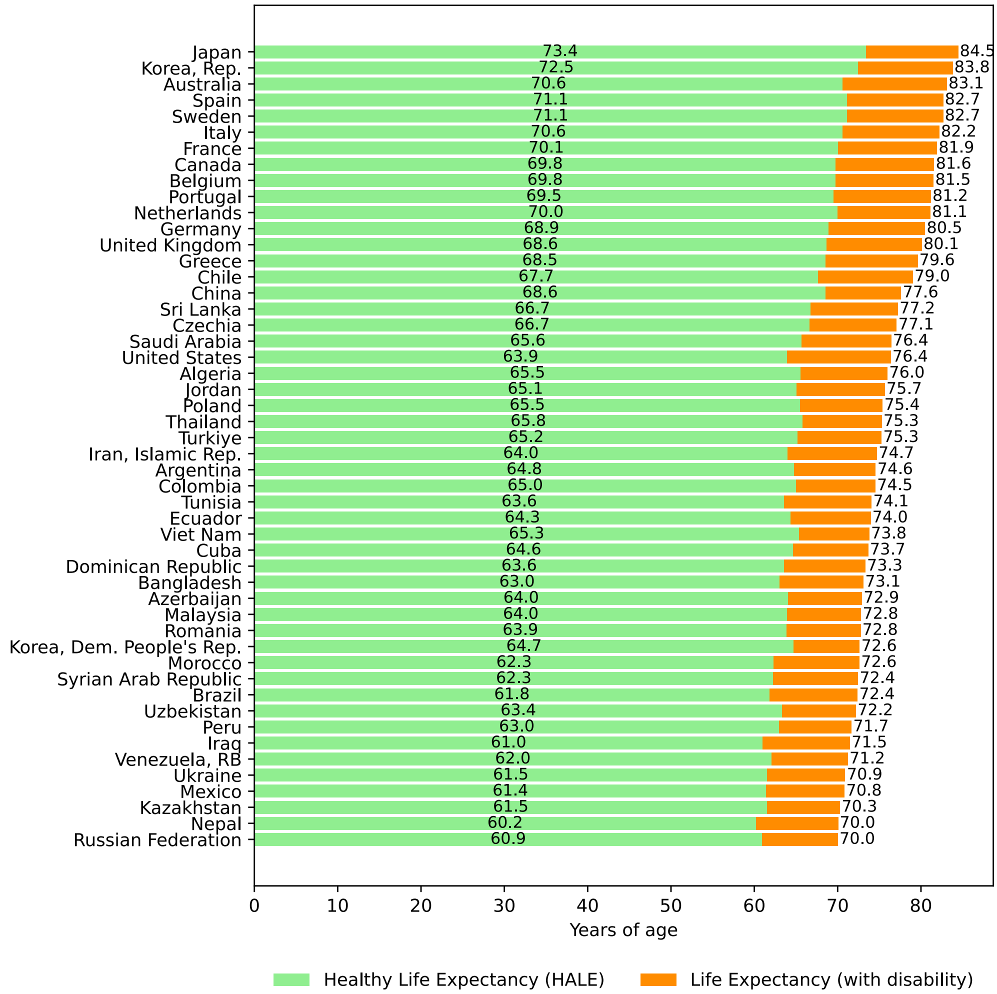


Figure 1: Life expectancy and healthy life expectancy (HALE) in countries with a population of more than 10 million in 2021. Data source: WHO.

[1]The recent Covid-19 pandemic is shown to have contributed to the increase in informal care intensity (Andrade et al., 2022; Kwon et al., 2024), with female caregivers being put under a higher burden of caregiving provision (Cohen et al., 2021), while depressing the public health care system (Susskind and Vines, 2020; Nguyen et al., 2025).

In most societies, caring for elderly parents is regarded as the responsibility of children. This obligation, however, imposes a substantial burden on subsequent generations. While siblings may share the task, each child is unavoidably responsible for parental care. By contrast, childcare burdens are adjustable, since parents can decide how many children to have. Faced with heavy parental care demands, adult children may reduce fertility, which lowers the ratio of working-age individuals to the elderly in the next generation and further intensifies the care burden. Confronted with heavier obligations, the next generation may also choose to have fewer children. In this way, increased longevity and caregiving responsibilities can trigger a cumulative decline in fertility across generations. To fully explain falling fertility, it is therefore essential to account for these intergenerational externalities.[2]

This paper investigates the implications of such externalities for population dynamics and other economic variables within an overlapping generations framework à la Diamond (1965). In our model, economic agents born to the same parent form sibling groups, which are solely responsible for caring for both their parents and their children while making all relevant decisions. Each group optimally allocates its members' time between market work, child-rearing, and eldercare to maximize overall welfare.

Childcare and eldercare need not be provided directly by group members; they may instead be outsourced to hired caregivers. Consequently, care arrangements can vary considerably: one group may allocate all labor to market work and purchase all necessary services, while another may assign some members to provide care internally, supported financially by others in market employment. Despite this heterogeneity, the model implies that consumption, savings, and fertility are determined at the same level across all groups.

By assuming that sibling groups act as decision-making units, our model avoids the free-rider problem in parental care. If each child decided independently, some would provide no care while benefiting from others efforts, resulting in insufficient overall support. To avoid such outcomes, parents might choose to have only one child, further depressing fertility. Collective decision-making eliminates this strategic problem, though fertility can still decline because groups base childbearing solely on their own costs and benefits, without considering effects on future generations.

The long-term state of the economy depends critically on the time intensity of both childcare and eldercare. When demands are modest, sibling groups can balance caregiving and work, allowing for "*sustainable growth*." By contrast, when caregiving time requirements exceed a threshold, the burden reduces fertility, which further increases the next generations obligations. Over time, the working population devotes nearly all of its time to caregiving, leaving little for productive work – what we call "*nursing hell*."

A child allowance policy can partially mitigate this outcome by encouraging higher fertility, increasing the number of siblings available to share parental care. A larger sibling group reduces the caregiving burden per worker, making it easier to sustain childbearing. However, when care requirements are extremely time-intensive, such policies may fail: within just a few generations, families may be unable to provide adequate care for both parents and children, leading to *structural collapse*.

[2] Although less pressing than eldercare demands, rising childcare time can create similar dynamics. Dotti Sani and Treas (2016) and Doepke et al. (2023) document this trend, especially among highly educated parents.

The demographic transition of declining fertility and population aging has been modeled in several ways. A common thread is the importance of the caregiving obligation toward elderly parents, but three distinct approaches explain why children provide such care. The first, examined by Wigger (1999), assumes a pure altruism where children derive utility from caregiving. The second, represented by Raut and Srinivasan (1994) and Chakrabarti (1999) , models caregiving as exchange, with children providing care in return for financial transfers. This category also includes bargaining models (Komura and Ogawa, 2018; Leroux and Pestieau, 2014; Yakita, 2024). The third approach views caregiving as a family duty imposed by social norms, without utility or compensation, as in Canta et al. (2016); Tran (2025) and Yakita (2023). Our model adopts this third perspective as it is widely supported by empirical evidence from Europe (Klimaviciute et al., 2017), China (Brasher, 2022), Thailand (Bui et al., 2026), and East and Southeast Asia (Chan, 2005).

Relative to this literature, our model introduces two key innovations. First, instead of focusing on inefficiencies in public LTC provision, as in Yakita (2023), we emphasize the demographic externalities of dual caregiving itself. Fertility decisions determine future caregiving burdens, which then feed back into fertility, potentially triggering cumulative declines across generations. This mechanism yields the novel possibility of a "nursing hell," where labor supply collapses under excessive caregiving demands. Second, our framework models sibling groups as joint decision-making units, eliminating free-rider problems and allowing explicit allocation of time between market work, childcare, and eldercare, while distinguishing skilled from unskilled labor. The result is a richer systemic analysis demonstrating that even with efficient LTC provision, demographic traps can emerge unless fertility-supporting measures are adopted. In this sense, our model complements and extends Yakita (2023)'s policy-oriented approach by showing that sustainability depends not only on public care efficiency, but also on demographic feedback loops linking fertility and caregiving.

The paper is organized as follows. Section 2 presents the model. Section 3 derives the equilibrium. Section 4 provides numerical illustrations. Section 5 shows an alternative case when the sibling group cares about the next generation. Section 6 analyzes the effects of child allowance policies. Section 7 discusses the robustness of assumptions on care burdens. Finally, section 8 concludes.

2 Model

Time is discrete, and the economy starts its operation in period 1 and continues indefinitely.

2.1 Production

In each period, a single final good is competitively produced using the following technology:

$$Y_t = A\left[K_t^{\alpha}(L_t^s)^{1-\alpha} + bL_t^u\right], \tag{1}$$

where Y_t, K_t, L_t^s, and L_t^u, respectively, denote the output of the final good, the input of physical capital, that of skilled labor, and that of unskilled labor; A, b, and α are constants satisfying $A > 0, b > 0$, and $\alpha \in (0, 1)$.

The production technology in (1) implies that physical capital and skilled labor are more complementary than physical capital and unskilled labor.[3] Under perfect competition, the factor prices are determined as

$$w_t^s = (1-\alpha)Ak_t^{\alpha}, \tag{2}$$

$$R_t = \alpha A k_t^{\alpha-1}, \tag{3}$$

$$w_t^u \geq Ab. \tag{4}$$

where $k_t \equiv K_t/L_t^s$; w_t^s, w_t^u and R_t, respectively, denote the wage rates of skilled and unskilled labor and the gross rental rate of capital. Note that unskilled labor is hired only when the equality in (4) is established.

2.2 Overlapping generations of agents

At the beginning of each period, a new generation of three-period-lived agents is born in the economy. They are divided into groups of siblings by who their birth parent is. In period 1, moreover, there are old agents of measure N_0, each of whom has $s_0 > 0$ units of capital and $n_0(> \underline{n})$ children, where the definition of $\underline{n}$ will be explained shortly.

Figure 2 visualizes the family structure. Agents born to the same parents form sibling groups, and these groups have the responsibility to care for their parents at the preceding node and the children at the succeeding nodes. Care services do not necessarily have to be provided by group members but rather by caregivers hired from the labor market.

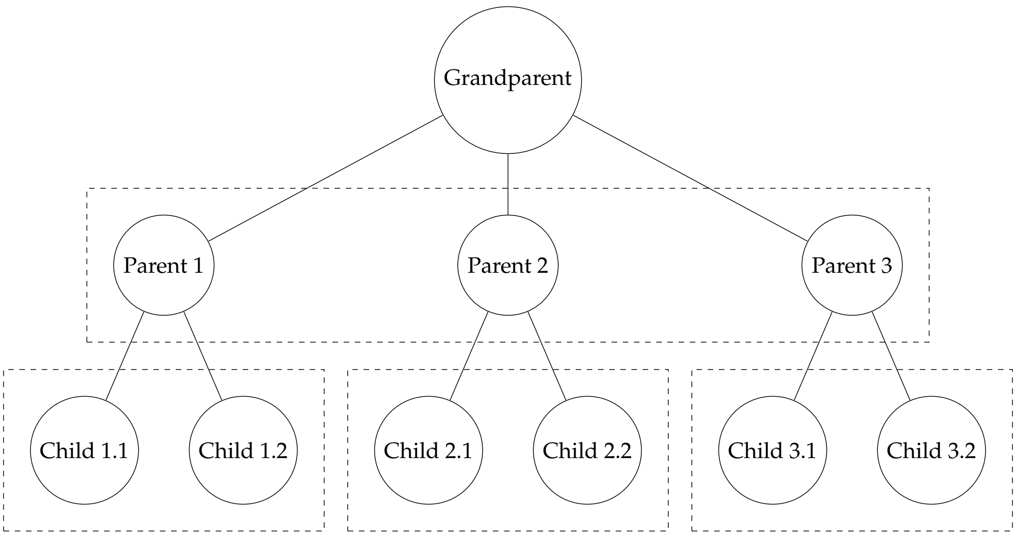


Figure 2: A visualization of a family structure with three generations.

Note: Agents enclosed by a dashed box belong to the same group and make collective decisions. For illustration purposes, each parent is assumed to have only two children, but the number can differ across generations. Although only three generations are shown here, the model features an infinite number of generations following the same structure.

[3] See Galor and Weil (1996); Kimura and Yasui (2007); Chen (2010); Day (2016) for other uses of the same technology. The empirical validity of capital-skill complementarity can be found in Duffy et al. (2004).

In the first period of life, the agents make no economic decisions. Decisions of great importance to a sibling group, such as the number of children, consumption, savings, and division of labor, are all made when economic agents are middle-aged. At that period, each group maximizes its overall welfare by optimally allocating its members' manpower to market work, child-rearing, which takes z units of time per child, and elderly care, which takes $\overline{x}$ units of time. In the third period, the agents consume all the goods they saved in the previous period and die.

Consider a group of siblings who reached middle age in period t. We assume that the utility function for each middle-aged sibling is given by[4]

$$u_t = \ln c_t + \gamma \ln(n_t - \underline{n}) + \beta \ln d_{t+1}. \tag{5}$$

where parameters β, γ are strictly positive, less than 1, weighing the tastes for consumption and children. The variables c_t, d_{t+1} and n_t represent the consumption level in period t, that in period $t+1$, and the number of children born and raised at time t. As a result, at each time t, there are n_{t-1} middle-aged adults in each sibling group.

The value of $\underline{n}$ is strictly positive and acts as a lower bound of fertility.[5] This parameter has an intuitive interpretation in our framework. Here, the middle-aged adults are the providers of care for elderly agents. Consequently, the first few children (some of whom will be carers in the next period) may be born out of their parents' necessities. After this "subsistence" fertility level is met, the decisions about having extra children now reflect the true preferences rather than needs. We further define $\underline{n}$ as given by the smaller root of the following quadratic equation:

$$zn^2 - n + \overline{x} = 0, \tag{6}$$

which implies that[6]

$$\underline{n} = \frac{1 - \sqrt{1 - 4\overline{x}z}}{2z}. \tag{7}$$

If households decide to have a fertility lower than $\underline{n}$, then the following holds for any n $(< \underline{n})$:

$$1 - zn - \overline{x}/n < 0.$$

As will be shown later, the left-hand side presents the effective labor time. This means that the total amount of labor allocated to the middle-aged generation will not be enough to meet the long-term needs of child and elder care.

For now, following (7), we assume that $\overline{x}$ and z satisfy:

$$\overline{x}z \leq 1/4.$$

[4] Note that we use a Stone-Geary preference for children, similar to previous works such as Voigtländer and Voth (2013); Black et al. (2013).

[5] Baudin et al. (2015) have introduced a similar innovation in modeling fertility preferences to allow for childlessness by setting $\underline{n}$ negative.

[6] This assumption is also convenient for analysis. The necessary condition for the model in this paper to have a steady state is

$$\underline{n} < (1 + \sqrt{1 - 4\overline{x}z})/2z,$$

which is obviously satisfied by this assumption.

so that $\underline{n}$ returns a positive number. The critical implications of $\underline{n}$ will be discussed in Section 7 when we relax this assumption.

As n_{t-1} is the population of a sibling group, we can index each member by a unique real number from 0 to n_{t-1}. In period t, they are endowed with one unit of time, which can be spent on training, work for income, childcare, and elder care. When working, each member has the choice of becoming either a skilled or an unskilled worker. To become a skilled worker, one must spend σ units of their endowed time on training. In contrast, no such training is required to become an unskilled worker.

The siblings collectively maximize the weighted sum of their utilities,

$$\int_0^{n_{t-1}} \omega_t(i)[\ln c_t(i) + \gamma \ln(n_t(i) - \underline{n}) + \beta \ln d_{t+1}(i)]di, \tag{8}$$

where $\omega_t(i)$, $c_t(i)$, $n_t(i)$ and $d_{t+1}(i)$, respectively, denote the weight, the middle-age consumption, the number of children, and the old-age consumption of the member indexed by $i \in [0, n_{t-1}]$, and the weights must satisfy

$$\omega_t(i) > 0 \quad \text{for } \forall i \in [0, n_{t-1}]$$

and

$$\int_0^{n_{t-1}} \omega_t(i)di = 1.$$

It is natural to assume that all members are given the same weight, or

$$\omega_t(i) = 1/n_{t-1} \quad \text{for } \forall i \in [0, n_{t-1}]. \tag{9}$$

If there is variation in the weights assigned to members, those assigned lower weights may leave the group, creating a discrepancy with the assumption that the group maximizes the weighted sum of all members' utilities. When (9) is true, the weights do not depend on indices, and thus maximizing (8) is reduced to maximizing

$$\int_0^{n_{t-1}} [\ln c_t(i) + \gamma \ln(n_t(i) - \underline{n}) + \beta \ln d_{t+1}(i)]di. \tag{10}$$

The maximization of (10) is achieved through the following three steps. First, the siblings determine who will become skilled workers and who will become unskilled workers. Second, given the number of skilled and unskilled workers determined in the first step, they decide what task each worker will perform. Finally, given the total income earned by skilled and unskilled workers, they determine each member's consumption for the current and next periods, as well as the number of children. It is well known that the outcome of such step-by-step decision-making can be obtained by working backward from the final step.

Let n^s_{t-1}, n^u_{t-1}, w^s_t, w^u_t, and h_t, respectively, denote the numbers of skilled and unskilled workers in this group, prices of skilled and unskilled labor, and total hours skilled workers are assigned to unskilled labor, such as caregiving. Using these notations, we can express the budget constraint of this group as

$$\int_0^{n_{t-1}} [c_t(i) + s(i) + w^u_t z n_t(i)]\, di + w^u_t \overline{x} = w^s_t[n^s_{t-1}(1-\sigma) - h_t] + w^u_t(n^u_{t-1} + h_t),$$

Savings are used for good consumption in old age:

$$d_{t+1}(i) = R_{t+1}s_t(i),$$

where R_{t+1} is the gross returns on the saved amount. The lifetime budget constraint can now be rewritten as

$$\int_0^{n_{t-1}} \left[c_t(i) + \frac{d_{t+1}(i)}{R_{t+1}} + w_t^u z n_t(i) \right] di + w_t^u \overline{x} = w_t^s \left[n_{t-1}^s (1-\sigma) - h_t \right] \\ + w_t^u \left(n_{t-1}^u + h_t \right). \tag{11}$$

The LHS of this equation summarizes the expenditures of this group. Specifically, $\int c_t(i)di$, $\int (d_{t+1}(i)/R_{t+1})di$, $\int w_t^u z n_t(i) di$, and $w_t^u \overline{x}$, respectively, represent this group's expenses for consumption, saving, childcare, and caring for their elderly parent. On the other hand, the RHS summarizes the incomes of this group. Specifically, its first and second terms, respectively, represent the compensation for providing skilled labor and the compensation for providing unskilled labor. Note that this budget constraint is unaffected by whether the siblings choose to care for their parent or children by themselves. If the siblings do not supply h_t units of their unskilled labor to the labor market, but instead use it to care for their parent and children, then this decision only reduces the values on both sides of the budget constraint by $w_t^u h_t$, but its equality still holds.

To analyze the final step, it is more convenient to rewrite the above budget constraint as follows:

$$\int_0^{n_{t-1}} \left[c_t(i) + \frac{d_{t+1}(i)}{R_{t+1}} + w_t^u z n_t(i) \right] di = I_t, \tag{12}$$

where

$$I_t \equiv w_t^s [n_{t-1}^s (1-\sigma) - h_t] + w_t^u (n_{t-1}^u + h_t) - w_t^u \overline{x}. \tag{13}$$

In the final step, we can treat the value of I_t as fixed, since the values of n_{t-1}^s and n_{t-1}^u are determined in the first step, and that of h_t is determined in the second step. To find the optimal values of $c_t(i)$, $d_{t+1}(i)$ and $n_t(i)$, we only need to maximize (10) subject to the budget constraint (12).

Given I_t, the optimal values of $c_t(i)$, $d_{t+1}(i)$ and $n_t(i)$ are determined as follows:

$$c_t(i) = \frac{I_t/n_{t-1} - w_t^u z \underline{n}}{1+\gamma+\beta}, \tag{14}$$

$$d_{t+1}(i) = \frac{\beta R_{t+1}(I_t/n_{t-1} - w_t^u z \underline{n})}{1+\gamma+\beta}, \tag{15}$$

$$n_t(i) - \underline{n} = \frac{\gamma (I_t/n_{t-1} - w_t^u z \underline{n})}{(1+\gamma+\beta) w_t^u z}. \tag{16}$$

As can be seen from these results, all members of this sibling group enjoy the same level of consumption and have the same number of children. This is a direct result of assigning identical weights to all members. Substituting (14) – (16) into (10) produces

$$n_{t-1} \left[(1+\gamma+\beta) \ln \frac{I_t/n_{t-1} - w_t^u z \underline{n}}{1+\gamma+\beta} + \gamma \ln \frac{\gamma}{w_t^u z} + \beta \ln \beta R_{t+1} \right],$$

which is the indirect utility of this group. Obviously, it is increasing with I_t, meaning that this group's optimization is reduced to maximizing (13) by optimally choosing the values of n^s_{t-1}, n^u_{t-1}, and h_t.

In the second step, the siblings seek the values of h_t that maximize (13), or solve the following maximization problem:

$$\max_{u_t \in [0, n^s_{t-1}(1-\sigma)]} w^s_t[n^s_{t-1}(1-\sigma) - h_t] + w^u_t(n^u_{t-1} + h_t) - w^u_t \overline{x},$$
$$n^s_{t-1} \text{ and } n^u_{t-1} \text{ given.}$$

The solution to this problem is determined as

$$h_t = \begin{cases} n^s_{t-1}(1-\sigma) & \text{if } w^s_t < w^u_t \\ \forall h \in [0, n^s_{t-1}(1-\sigma)] & \text{if } w^s_t = w^u_t \\ 0 & \text{if } w^s_t > w^u_t \end{cases} \tag{17}$$

This result implies that it is impossible for $w^s_t < w^u_t$ in equilibrium. In that case, all skilled workers choose to engage in unskilled labor, which eliminates the aggregate supply of skilled labor completely. Since skilled labor is essential for economic activity in this model, this elimination makes it impossible for the economy to achieve equilibrium. When $w^s_t \geq w^u_t$, Eq.(17) implies that the following is true:

$$(w^s_t - w^u_t)h_t = 0,$$

which reduces (13) to

$$I_t = w^s_t n^s_{t-1}(1-\sigma) + w^u_t n^u_{t-1} - w^u_t \overline{x}. \tag{18}$$

Finally, in the first step, the siblings seek the pairs of n^s_{t-1} and n^u_{t-1} that maximize (18). Since $n^s_{t-1} + n^u_{t-1} = n_{t-1}$, their maximization problem can be expressed as

$$\max_{n^s_{t-1} \in [0, n_{t-1}]} w^s_t n^s_{t-1}(1-\sigma) + w^u_t(n_{t-1} - n^s_{t-1}) - w^u_t \overline{x},$$

the solution to which is

$$n^s_{t-1} = \begin{cases} 0 & \text{if } (1-\sigma)w^s_t < w^u_t \\ \forall n^s \in [0, n_{t-1}] & \text{if } (1-\sigma)w^s_t = w^u_t \\ n_{t-1} & \text{if } (1-\sigma)w^s_t > w^u_t \end{cases} \tag{19}$$

Eqs.(17) and (19) jointly imply that the following must hold in equilibrium:

$$(1-\sigma)w^s_t = w^u_t, \tag{20}$$

which reduces (18) to

$$I_t = w^u_t(n_{t-1} - \overline{x}) = (1-\sigma)w^s_t(n_{t-1} - \overline{x}). \tag{21}$$

If $(1-\sigma)w^s_t > w^u_t$, all middle-aged agents choose to become skilled workers (see (19)). In addition, they never supply the unskilled labor, since $(1-\sigma)w^s_t > w^u_t$ implies $w^s_t > w^u_t$ (see (17)). As a result, the aggregate supply of unskilled labor becomes zero. If

$(1-\sigma)w_t^s < w_t^u$, all middle-aged agents choose to become unskilled workers (see (19)), which would eliminate the supply of skilled labor completely. Only when $(1-\sigma)w_t^s = w_t^u$, skilled and unskilled workers can coexist, supplying two types of labor essential to economic activity. Since Eq.(20) implies $w_t^s > w_t^u$, skilled workers specialize in skilled labor, while unskilled workers specialize in unskilled labor, according to (17). This can be regarded as achieving a division of labor based on comparative advantage within the sibling group.

Substituting (20) and (21) into (14)-(16), we obtain the following optimal choices:

$$c_t = \frac{1}{1+\gamma+\beta} w_t^s (1-\sigma)\left(1 - z\underline{n} - \frac{\overline{x}}{n_{t-1}}\right), \tag{22}$$

$$d_{t+1} = \frac{\beta}{1+\gamma+\beta} R_{t+1} w_t^s (1-\sigma)\left(1 - z\underline{n} - \frac{\overline{x}}{n_{t-1}}\right), \tag{23}$$

$$n_t - \underline{n} = \frac{\gamma}{(1+\gamma+\beta)z}\left(1 - z\underline{n} - \frac{\overline{x}}{n_{t-1}}\right). \tag{24}$$

Note that compared to (14)-(16), the indices representing individuals have been removed from (22)-(24). This reflects the result that consumption levels and the number of children are identical within a sibling group.

Since all members of the sibling group attain the same level of consumption and number of children, and since that $(1-\sigma)w_t^s = w_t^u$ holds in equilibrium, it is more convenient to recast the household's problem in per-member terms. Using (11) and (21), the budget constraint of a sibling group can be rewritten as

$$n_{t-1}\left(c_t + \frac{d_{t+1}}{R_{t+1}} + w_t^u z n_t\right) = (1-\sigma)w_t^s(n_{t-1} - \overline{x}).$$

Dividing both sides by n_{t-1} and using the fact that $(1-\sigma)w_t^s = w_t^u$, we obtain the budget in per-member terms

$$c_t + \frac{d_{t+1}}{R_{t+1}} = (1-\sigma)w_t^s\left(1 - z n_t - \frac{\overline{x}}{n_{t-1}}\right). \tag{25}$$

The household's problem can now be recast as

$$\max_{c_t, d_{t+1}, n_t} \ln c_t + \gamma \ln(n_t - \underline{n}) + \beta \ln d_{t+1}$$

subject to (25). As expected, it yields the exact solutions as in (22) – (24).[7] In other words, the optimal choices made collectively by the sibling group are consistent with what each individual member would have chosen independently.

[7] The FOCs are

$$(n_t): \quad \frac{\gamma}{n_t - \underline{n}} = \frac{z(1+\beta)}{\left(1 - z n_t - \frac{\overline{x}}{n_{t-1}}\right)}, \tag{26}$$

$$(c_t): \quad c_t = \frac{(1-\sigma)w_t^s(1 - z n_t - \overline{x}/n_{t-1})}{1+\beta},$$

$$(d_{t+1}): \quad d_{t+1} = \beta R_{t+1} c_t.$$

We can use (7) to recover the solutions in (22) – (24).

The argument so far has focused solely on one sibling group. However, we can even argue that all sibling groups in period t have the same number of members (i.e., n_{t-1}), and thus that Eqs.(22)-(24) represent the consumption and number of children for any middle-aged agent in the period t. To see this point, let us first recall that at the beginning of this subsection, we made the assumption that in period 1, there are old agents of measure N_0, each of whom has n_0 ($> \underline{n}$) children, which is equivalent to that in that period, there are N_0 sibling groups, each of which contains n_0 members. Tracing the ancestry of sibling groups existing in period t will lead to one of these N_0 sibling groups. Next, let's focus on (24). This equation uniquely determines the path of sibling group sizes (i.e., $\{n_t\}_{t=1}^{\infty}$), given its initial value.

We can use (6) to rewrite (24) as

$$n_t - \underline{n} = \frac{\gamma \overline{x}}{(1+\gamma+\beta)z\underline{n}} \cdot \frac{n_{t-1} - \underline{n}}{n_{t-1}}, \tag{27}$$

implying that $n_t > \underline{n}$ for any $t \geq 1$ if $n_0 > \underline{n}$.[8] As we have seen, since the size of all N_0 sibling groups existing in period 1 is n_0, it follows from (24) or (27) that the sizes of all sibling groups in period t will also be identical, implying that consumption and the number of children for middle-aged agents in period t are identical regardless of the sibling group to which they belong.

While the middle-aged agents in period t enjoy the same levels of c_t, d_{t+1}, and n_t, there can be significant variation among sibling groups in career choices and approaches to caregiving. In one group, for example, all members may supply skilled labor to the labor market and purchase all childcare and eldercare services through the market. In another group, all necessary childcare and eldercare within the group may be handled by a few members, with their consumption potentially covered by other members who supply skilled or unskilled labor to the labor market. Such variation can arise because, when (20) is true, occupational choices within a group are not uniquely determined (see (19)), and because in this model, engaging in caregiving and supplying unskilled labor are essentially the same thing.

3 Equilibrium

Let N_t be the population of the middle-aged agents in period t. Their dynamics are given by the following equation

$$N_t = n_{t-1} N_{t-1}. \tag{28}$$

Define the skilled worker ratio as

$$\phi_t = \frac{N_t^s}{N_t}.$$

Since a period in the model is long, capital is assumed to depreciate fully after one period. The supply of capital is given by:

$$K_t = N_{t-1} s_{t-1}. \tag{29}$$

[8] This result is noteworthy, since the number of children cannot yield any utility unless it exceeds $\underline{n}$ (see (5)).

The supply of skilled labor to the market is

$$L_t^s = (1-\sigma)\phi_t N_t.$$

We can then express $k_t (\equiv K_t/L_t^s)$ as

$$k_t = \frac{s_{t-1}N_{t-1}}{(1-\sigma)\phi_t N_t} = \frac{s_{t-1}}{(1-\sigma)\phi_t n_{t-1}}, \tag{30}$$

where the second equality is obtained from (28). Given n_{t-1} and s_{t-1}, the values of k_t and ϕ_t are determined as follows.

When $(1-\sigma)w_t^s = w_t^u > Ab$, firms are not willing to hire unskilled workers (see (4)). These workers are all either providing care within the family or selling the care services in the market. The remaining skilled workers will supply skilled labor to the production sector. This implies that

$$\phi_t = 1 - zn_t - \frac{\overline{x}}{n_{t-1}}. \tag{31}$$

Note that $(1-\sigma)w_t^s > Ab$ is equivalent to

$$k_t > \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}. \tag{32}$$

Using (30) to eliminate k_t from this inequality, we obtain

$$\frac{s_{t-1}}{(1-\sigma)\phi_t n_{t-1}} > \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}$$

or

$$\left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha} \frac{s_{t-1}}{(1-\sigma)n_{t-1}} > \phi_t.$$

Thus, in this case, the following must hold:

$$\left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha} \frac{s_{t-1}}{(1-\sigma)n_{t-1}} > 1 - zn_t - \frac{\overline{x}}{n_{t-1}} \quad (=\phi_t). \tag{33}$$

When $(1-\sigma)w_t^s = w_t^u = Ab$, firms may hire some unskilled workers. Note that $(1-\sigma)w_t^s = Ab$ is equivalent to

$$k_t = \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}.$$

Use (30) to eliminate k_t from this equation. Then

$$\frac{s_{t-1}}{(1-\sigma)\phi_t n_{t-1}} = \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}$$

or

$$\phi_t = \left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha} \frac{s_{t-1}}{(1-\sigma)n_{t-1}}. \tag{34}$$

From (33) and (34), we can say that the value of ϕ_t is determined as

$$\phi_t = \min\left\{ \left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha} \frac{s_{t-1}}{(1-\sigma)n_{t-1}}, 1 - zn_t - \frac{\overline{x}}{n_{t-1}} \right\}. \tag{35}$$

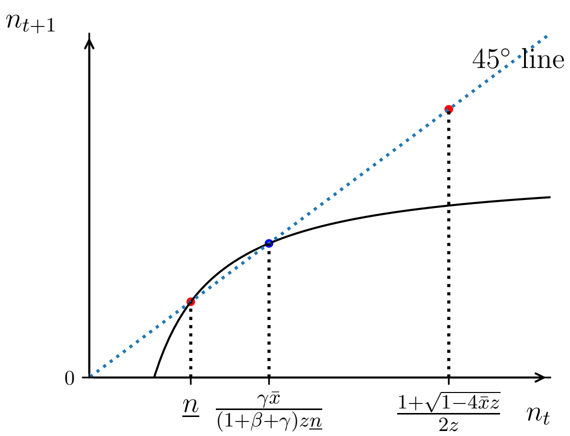
Substituting this value into (30) also yields the value of k_t.

Definition 1. The equilibrium of this economy is the sequence of endogenous variables $\{(c_t, d_t, n_t, \phi_t, k_t, w_t^s)\}_{t=1}^{\infty}$ that is uniquely determined by (22), (23), (27), (30), (35), and (2), given the initial values $n_0 \in (\underline{n}, \frac{1+\sqrt{1-4\overline{x}z}}{2z})$ and $s_0 > 0$.

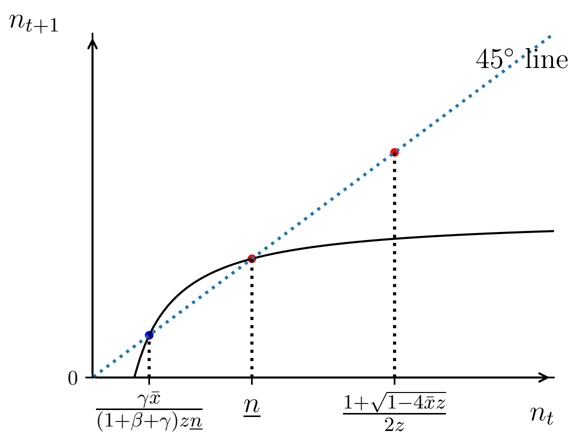
Then, we can state the following.

Proposition 1. *Define η as $\eta \equiv \gamma(1+\gamma+\beta)/(1+2\gamma+\beta)^2$. Then the equilibrium path of n_t converges to $\gamma\overline{x}/[(1+\gamma+\beta)z\underline{n}]$ if $\overline{x}z \in (0, \eta)$, and to $\underline{n}$ if $\overline{x}z \in [\eta, 1/4]$.*

Proof. See Appendix A. □



(a) When $\overline{x}z \in (0, \eta)$



(b) When $\overline{x}z \in [\eta, 1/4]$

Figure 3: The dynamics of n_t when $\overline{x}z \in (0, 1/4]$.

As visualized in Figure 3, the dynamics of fertility always converge to a higher equilibrium point. However, there is a considerable difference between what happens in equilibrium in the case of convergence to $\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}$ (left panel) and in the case of convergence to $\underline{n}$ (right panel). On the left panel, the agents in this economy are put under sufficiently low dual caregiving pressure. Thus, the economy has enough manpower and resources to allocate people into working and caring, which we can use the word "sustainable growth" to describe its long-run trajectory. In this economy, agents born in the distant future enjoy a decent level of consumption. However, their counterparts on the right-hand side economy can consume almost nothing (see (22)-(24)). In the long run of the second case, almost all of the endowed labor is spent on childcare and caring for elderly parents, with little available for income-generating labor. As will be shown later in the next section, this economy can be trapped in a continuous, multi-generational decline in fertility and welfare, a condition best described as a *nursing hell* since almost all resources are consumed by dual caregiving duties.

4 Numerical examples

This section provides some numerical examples to illustrate the above contrasting dynamics. The model is calibrated for the US economy, under the assumption that one period in the model is equivalent to 30 years in real life.

Following the real business cycle literature, we set the subjective discount to $\beta = 0.99^{120} = 0.3$ (equivalent to a discount rate of 0.99 per quarter) and the capital share $\alpha = 0.36$. According to Squicciarini et al. (2015), the training time for a typical skilled worker varies with countries. For example, the training time for a newly hired employee is 0.51 hours per day in the US (Loewenstein and Spletzer, 2000), 0.89 hours per day in Japan (Kurosawa, 2001), or 1.56 hours per day for all employees in the Netherlands (Nelen and De Grip, 2009). Averaging these numbers results in approximately 1 hour per day for training, or 12% of an 8-hour working time. Aside from on-the-job training, we must also take into account tertiary education, which typically lasts 4 years, or is equivalent to 13.3% of the total time endowment for an adult in the model. Summing up the education and on-the-job training time, we calibrate $\sigma = 0.25$.

The child-rearing time cost is crucial in determining the dual caregiving severity. Dotti Sani and Treas (2016), using a global dataset (largely pre–2006), report that mothers and fathers aged 18–65 spend on average 36.5 minutes per day on childcare across the sampled countries, which would result in $z = 0.076$. This is consistent with the value de la Croix and Doepke (2003) used in their calibration ($z = 0.075$) for the US. However, recent evidence shows that there has been an upward trend in parental time devoted to children in recent years (Cardia and Gomme, 2018). Specifically, Doepke et al. (2023) show that in the US (2019), non-employed mothers and fathers, respectively, spend 162 minutes and 94 minutes a day on childcare, while full-time working mothers and fathers spend 84 and 57 minutes. Because the model implies that child-rearing can be either self-provided or bought from the market, the time expenditure of non-employed parents provides a good approximation of the total opportunity costs of childcare. The total amount of minutes spent for non-employed parents is 256 minutes per day, or 26.67% of the total time endowment.[9] If we assume that the children live with parents until the age of 18, then one can compute $z = 0.26 \times 18/30 = 0.16$.

In what follows, we choose appropriate values for elderly caregiving, alongside other relevant parameters. When $\overline{x}z \leq 1/4$, the model distinguishes two distinct growth patterns, depending on whether $\overline{x}z$ is larger or smaller than the variable η, which depends on $\beta, \gamma, \overline{x}$, and z. Our goal in this example is to assign values to $\overline{x}$ to see the trajectories of the realized growth scenarios. Canta et al. (2016) report that in Europe, about 40% of the people aged 65 or older report having functional limitations that require assistance, while Yakita (2023) cited that one third of the middle-aged parents with university-aged children in Japan (2018) have experience caring for both their own children and their elderly parents simultaneously (more than half if we consider working female only (Kawabe et al., 2024) in the 2019–2020 sample). In the US, the number is between 20.8% (Suh, 2016), 24.3% (Lei et al., 2023) and 36% for married adults (Parker and Patten, 2013). We thus assign $\overline{x} = 1/3$. Under this configuration, the economy is on a "sustainable growth" path.

Following the calibration so far, we need to assign a value to γ that governs altruism. In the US, as well as other advanced economies, the fertility rates are very close

[9] Assume that a person works 8 hours a day, or 480 minutes, then the opportunity costs of child-rearing amount to $256/(480 \times 2) = 26.67\%$

to the reproduction level. To arrive at this number, we set $\gamma = 0.27$, which is similar to de la Croix and Doepke (2003). As a result, the parameter $\eta = 0.1252$. Taking this number as a benchmark, we can generate the case of "nursing hell" growth trajectory by setting $\bar{x}$ at a sufficiently high level. In this case, setting $\bar{x} > 0.7825$ will do the job, so we set $\bar{x} = 0.8$ for a counter-factual simulation.

For other scaling parameters, as they only affect the division of labor, not the dynamics that we are interested in (see (27)).[10] Thus, we set $b = 0.2$ and $A = 6$ so that in the long run, about half of the population is skilled (Prettner and Strulik, 2020). In what follows, we will generate the growth patterns of the economy, whose only difference is the elderly care needs $\bar{x}$. Initial values are the same, where $s_0 = 0.1$ and $n_0 = 3$, so both economies start off with low wealth and high fertility. The results are present in Table 1 and 2, with the last row of each table showing the steady-state outcome.

Table 1: Sustainable growth when $\bar{x}z < \eta$.

t	n_t	s_t	c_t	ϕ_t	k_t	u_t
0	3.0000	0.1000				
1	1.2464	0.1909	0.6378	0.5058	0.0879	-0.5503
2	1.0800	0.2592	0.8657	0.5624	0.3632	-0.3065
3	1.0361	0.2928	0.9780	0.5287	0.6052	-0.2000
4	1.0222	0.3065	1.0237	0.5180	0.7274	-0.1588
5	1.0175	0.3117	1.0412	0.5144	0.7772	-0.1433
6	1.0160	0.3136	1.0476	0.5131	0.7960	-0.1375
7	1.0154	0.3144	1.0500	0.5127	0.8028	-0.1354
⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	1.0151	0.3148	1.0514	0.5125	0.8067	-0.1342

Parameters: $\bar{x} = 0.33, \beta = 0.99^{120}, \gamma = 0.271, \alpha = 0.36, \sigma = 0.25, z = 0.16, A = 6, b = 0.2$ so that $\bar{x}z = 0.058 < \eta = 0.125$.

Table 1 reports the equilibrium path of the economy when $\bar{x}z < \eta$. As predicted by Proposition 1, the fertility rate decreases over time and stabilizes at 1.0151 at the steady state. Although there is almost no population growth, the pressure of dual caregiving is sufficiently low, which allows agents to participate in the workforce. Note that in this economy, all unskilled labor is dedicated to caregiving duties. The remaining agents, about 51% of the population, acquire training and supply skilled labor to the production sector. As everything functions properly, the economy is said to be on a *sustainable growth* path.

In contrast, Table 2 depicts a widely different picture. Here, the care burden $\bar{x}z$ exceeds the threshold η, driving fertility towards 0.94 in the long run, which even suggests a declining population in absolute level.[11] Such a level of fertility is too low to sustain the economy under the intense pressure from dual caregiving. The economy's labor force is almost entirely dedicated to caregiving, which decimates the supply of

[10] Appendix C details the division of labor: under which condition of A and b would result in unskilled labor being spent fully on caregiving or only partially.

[11] The argument still stands without population decline, i.e., for $\underline{n} \geq 1$.

Table 2: Nursing hell growth when $1/4 > \bar{x}z > \eta$.

t	n_t	s_t	c_t	ϕ_t	k_t	u_t
0	3.0000	0.1000				
1	1.5684	0.1357	0.4531	0.4824	0.0921	-1.1147
2	1.3067	0.1353	0.4518	0.2809	0.4106	-1.3676
3	1.1969	0.1147	0.3833	0.1963	0.7032	-1.7154
4	1.1365	0.0939	0.3135	0.1498	0.8535	-2.0627
5	1.0983	0.0773	0.2583	0.1203	0.9151	-2.3781
6	1.0719	0.0649	0.2168	0.1001	0.9384	-2.6577
7	1.0527	0.0555	0.1853	0.0852	0.9469	-2.9056
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	0.9433	0.0007	0.0022	0.0010	0.9517	-9.8360

Parameters: $\bar{x} = 0.8, \beta = 0.99^{120}, \gamma = 0.271, \alpha = 0.36, \sigma = 0.25, z = 0.16, A = 6, b = 0.2$ so that $\bar{x}z = 0.128 > \eta = 0.125$.

skilled labor essential for production. The resulting shortage of skilled workers cripples economic output, which in turn causes consumption and savings to become extremely small. Ultimately, as consumption approaches zero, the economy will cease to function.[12]

5 An economy with altruistic sibling groups

One reason why phenomena such as "nursing hell" emerge in the economy examined so far is that each generation, when deciding how many children to have, completely disregards the effect of its decision on the next generation. If parents were to consider that having one more child could reduce the caregiving burden faced by their own children, the outcome might change significantly. To explore this point, this section analyzes an economy populated by sibling groups that exhibit altruism toward the next generation.[13]

Consider a group of siblings who reach middle age in period t. Their utility function is given by

$$U_t = \ln c_t + \gamma \ln(n_t - \underline{n}) + \beta \ln d_{t+1} + \delta U_{t+1}, \tag{36}$$

where $\delta \in (0, 1)$. Comparing this with (5) reveals that the lifetime utility of the sibling group, U_t, depends not only on their own consumption (c_t and d_{t+1}) and number of children (n_t), but also on the lifetime utility of their children (U_{t+1}). The final term on the right-hand side distinguishes this model from those in previous sections.

Each sibling group maximizes (36) subject to the budget constraint (25), yielding the following first-order condition for the number of births:

$$\frac{\gamma}{n_t - \underline{n}} + \delta \frac{1+\beta}{1 - zn_{t+1} - \bar{x}/n_t} \cdot \frac{\bar{x}}{n_t^2} = \frac{z(1+\beta)}{1 - zn_t - \bar{x}/n_{t-1}}. \tag{37}$$

[12] For example, setting $z = 0.22$ will collapse the economy after 57 periods.

[13] This section summarizes the results without formal proofs. For detailed derivations of the FOCs and the construction of the phase portrait, see the Appendix D.

This condition implies that, for a given n_{t-1}, this model generates a larger number of children (a higher n_t) than in the previous framework. When $\delta = 0$, eq.(37) reduces to (26). Comparing the two, the right-hand side—representing the marginal cost of child-rearing—remains the same, while the left-hand side—representing the marginal benefit—is higher in (37) due to the additional altruistic term. Since the right-hand side is an increasing function of n_t, this means that the equilibrium n_t must be higher when altruism is present.

The dynamical system (37) has two positive steady-state values: $\underline{n}$ and

$$n^* = \frac{\gamma\overline{x}/\underline{n} + \sqrt{(\gamma\overline{x}/\underline{n})^2 + 4z\delta\overline{x}(1+\beta+\gamma)(1+\beta)}}{2z(1+\beta+\gamma)}. \tag{38}$$

The existence of $\underline{n}$ indicates that even if more children are born, this economy may not necessarily escape from the nursing hell. As long as $\underline{n}$ is a steady-state value, one cannot rule out the possibility that the birth rate trajectory converges to it. Note also that n^* is meaningful only when $n^* > \underline{n}$, since utility arises only when births exceeds $\underline{n}$. When $\overline{x}z \in (0, \eta]$, we can state that $n^* > \underline{n}$, because the following inequalities hold:

$$\underline{n} \leq \frac{\gamma\overline{x}}{(1+\beta+\gamma)z\underline{n}} < n^*,$$

where the first inequality follows from $\overline{x}z \leq \eta$, and the second from (38).

When $\overline{x}z \in [\eta, 1/4]$, $n^* > \underline{n}$ if and only if δ satisfies

$$\underline{\delta}(\overline{x}z) < \delta < 1, \tag{39}$$

where

$$\underline{\delta}(\overline{x}z) \equiv \frac{1+\gamma+\beta}{1+\beta} \cdot \frac{1-\sqrt{1-4\overline{x}z}}{2\overline{x}z} - \frac{1+2\gamma+\beta}{1+\beta}.$$

Simple algebra shows that $\underline{\delta}(\overline{x}z)$ is a continuous, increasing function of $\overline{x}z$, with $\underline{\delta}(\eta) = 0$ and $\underline{\delta}(1/4) = 1$.

Indeed, the equilibrium path of n_t converges to n^* when $\overline{x}z \leq \eta$ or when $\overline{x}z \in (\eta, 1/4)$ and $\delta \in (\underline{\delta}(\overline{x}z), 1)$; it converges to $\underline{n}$ when $\overline{x}z = 1/4$ or when $\overline{x}z \in (\eta, 1/4)$ and $\delta \in (0, \underline{\delta}(\overline{x}z)]$. These outcomes can be verified by drawing phase diagrams.

Define l_t as

$$l_t \equiv 1 - zn_t - \overline{x}/n_{t-1}, \tag{40}$$

which represents the proportion of time devoted to income-earning activities and must be positive in equilibrium. Using this notation, we can rewrite (37) as

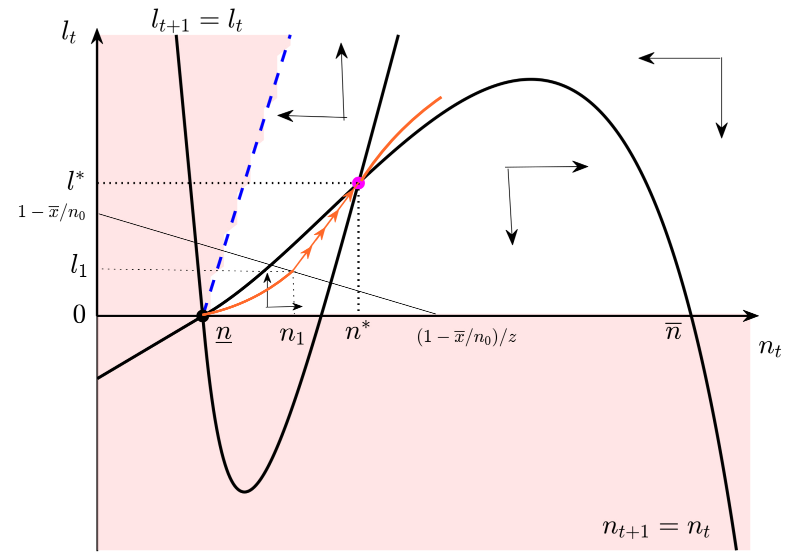
$$l_{t+1} = \frac{\delta(1+\beta)\overline{x}(n_t - \underline{n})l_t}{n_t^2[(1+\beta)z(n_t-\underline{n}) - \gamma l_t]}. \tag{41}$$

From (40) and (41), we can also obtain

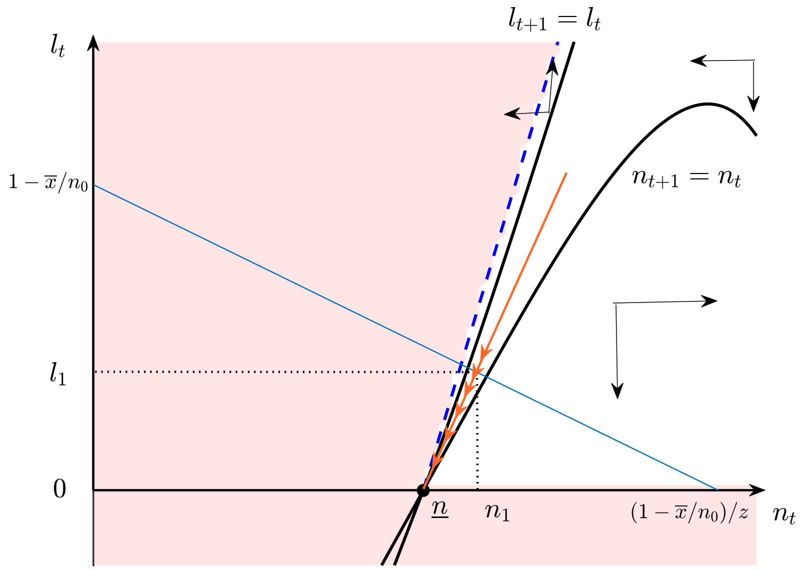
$$n_{t+1} = \frac{1}{z}\left\{1 - \frac{\overline{x}}{n_t} - \frac{\delta(1+\beta)\overline{x}(n_t-\underline{n})l_t}{n_t^2[(1+\beta)z(n_t-\underline{n}) - \gamma l_t]}\right\}. \tag{42}$$

Eqs.(41) and (42) describe the system's motion over time.[14]

[14] Appendix D details the construction of the phase portrait.



(a) Convergence to n^*.



(b) Convergence to $\underline{n}$.

Figure 4: Phase portrait for the system (41) and (42), excluding points in the red region, which are infeasible.

Figure 4a illustrates a typical phase diagram when $\overline{x}z \leq \eta$ or $\overline{x}z \in (\eta, 1/4)$ with $\delta \in (\underline{\delta}(\overline{x}z), 1)$. In these cases, the system has two steady states: $(\underline{n}, 0)$ and (n^*, l^*). As shown, $(\underline{n}, 0)$ is a source with no convergence path leading to it, while (n^*, l^*) is a saddle point. If the intersection of the line $l = 1 - zn - \overline{x}/n_0$ and the stable arm is denoted by (n_1, l_1), then (n_t, l_t) converges to (n^*, l^*). Thus, when the care burden is moderate—or even heavy but coupled with sufficient altruism toward children—the economy avoids falling into nursing hell.

By contrast, Figure 4b depicts the case where $\overline{x}z = 1/4$ or $\overline{x}z \in (\eta, 1/4)$ with $\delta \in (0, \underline{\delta}(\overline{x}z)]$. Here, the unique steady state is $(\underline{n}, 0)$, which is a saddle point with a single stable arm leading to it. The point (n_t, l_t) therefore converges to $(\underline{n}, 0)$. When the burden of caregiving is severe and parents are largely indifferent to their children's utility, the economy inevitably falls into a long-term nursing hell. Particularly when $\overline{x}z = 1/4$, the burden is so heavy that, regardless of parental altruism, the economy cannot avoid this outcome.

In summary, even under a heavy caregiving burden, the economy can avoid a nursing hell if parents care sufficiently about the welfare of their children—except in the extreme case of $\overline{x}z = 1/4$. The next section explores whether policy interventions can induce such altruistic motives among agents who are otherwise indifferent to their offspirng's utility.

6 Effects of child allowances

The previous section has shown that the right choice of fertility is critical when the dual caregiving burden is sufficiently large ($\overline{x}z > \eta$). We also know that some degree of inter-generational altruism can incentivize households to raise fertility, and therefore secure enough manpower for production. The question is how to design a system that encourages more childbearing.

As it turns out, this problem is similar to that of a Pigouvian subsidy that should be used to internalize the positive externality (by having more children). The government can intervene and apply this strategy by engineering a pronatalist policy. Let us consider a simple child allowance program that can be used to foster childbearing. Each sibling group can now receive a child allowance of $\theta > 0$ per birth. To finance this, the government imposes a tax rate of $\tau_t \in (0, 1)$ every period t on the household's wage income.

Under the new environment, the household problem has now become

$$\max_{c_t, n_t, d_{t+1}} \ln c_t + \gamma \ln(n_t - \underline{n}) + \beta \ln d_{t+1}$$

with respect to the new constraints

$$\begin{aligned} c_t + s_t &= (1-\tau_t)(1-\sigma)w_t^s \left(1 - zn_t - \frac{\overline{x}}{n_{t-1}}\right) + \theta n_t, \\ d_{t+1} &= R_{t+1} s_t. \end{aligned}$$

Note that this tax policy will not change the wage equilibrium condition because the tax is applied as a proportional rate on all wage income, regardless of whether it's from skilled or unskilled labor. Specifically, it still holds that $(1-\sigma)w_t^s = w_t^u$.[15] To close the model, the government balances its budget every period, such that

$$\tau_t = \frac{\theta n_t}{(1-\sigma)w_t^s \left(1 - zn_t - \frac{\overline{x}}{n_{t-1}}\right)}. \tag{43}$$

[15] Technically, it is $(1-\tau_t)(1-\sigma)w_t^s = (1-\tau_t)w_t^u$.

The household's optimal solutions are now given by

$$c_t = \frac{1}{1+\gamma+\beta}\left[(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)+\theta\underline{n}\right], \tag{44}$$

$$s_t = \frac{\beta}{1+\gamma+\beta}\left[(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)+\theta\underline{n}\right], \tag{45}$$

$$n_t-\underline{n} = \frac{\gamma}{1+\gamma+\beta}\frac{(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)+\theta\underline{n}}{(1-\tau_t)(1-\sigma)w_t^s z-\theta}. \tag{46}$$

Proposition 2. *If $\theta > 0$ and $n_0 > \underline{n}$, then $n_t > \underline{n}$ for all t.*

Proof. See Appendix B. □

This result implies that once the child allowance θ is implemented, starting from any initial value of fertility higher than $\underline{n}$, the fertility choice n_t is always greater than $\underline{n}$. Thus, the *nursing hell* problem can then be alleviated. Given $\delta > \underline{\delta}$, the government can choose an appropriate child allowance level θ. In the steady state, by replacing (37) with (46), we can establish a link between δ and θ:

$$\theta = \frac{(1-\sigma)w^s(1-zn^*-\frac{\overline{x}}{n^*})[z(n^*-\underline{n})(1+\beta+\gamma)-\gamma(1-z\underline{n}-\frac{\overline{x}}{n^*})]}{(n^*-\underline{n})(1+\beta)(1-\frac{\overline{x}}{n^*})}.$$

That is, for each level of δ satisfying (39), the above equation can be used to pin down δ for any given value of θ. Thus, it is more convenient to concentrate on the size and effects of θ. Although the positive effect of the child allowance policy on steady-state fertility is straightforward, it is equally important to examine its impact on welfare along the transition path.

To this end, we maintain the rest of the model structure unchanged, with production specified by (1). The wage for skilled labor is still

$$w_t^s = (1-\alpha)Ak_t^\alpha$$

where $k_t \equiv K_t/L_t^s$. The laws of motion for the middle-aged population and capital per skilled labor are given by

$$N_t = n_{t-1}N_{t-1}, \tag{47}$$

$$k_t = \frac{s_{t-1}}{(1-\sigma)\phi_t n_{t-1}}. \tag{48}$$

Given n_{t-1} and s_{t-1}, the rest of the model is determined as follows.

When $(1-\sigma)w_t^s = w_t^u = Ab$, some portion of unskilled labor will be allocated to final good production. This implies

$$w_t^s = (1-\alpha)Ak_t^\alpha = \frac{Ab}{(1-\sigma)}. \tag{49}$$

This implies

$$k_t = \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}.$$

Using (48) to eliminate k_t from this equation, we obtain the skilled worker ratio

$$\phi_t = \left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha} \frac{s_{t-1}}{(1-\sigma)n_{t-1}}.$$

On the other hand, when $(1-\sigma)w_t^s = w_t^u > Ab$, unskilled workers are not hired in the final good production and thus spend all their time on caring activities. This implies

$$\phi_t = 1 - zn_t - \frac{\overline{x}}{n_{t-1}}.$$

Using this information, the capital-skilled labor ratio becomes

$$k_t = \frac{s_{t-1}}{(1-\sigma)\left(1 - zn_t - \frac{\overline{x}}{n_{t-1}}\right) n_{t-1}} = \frac{s_{t-1}}{(1-\sigma)(n_{t-1} - zn_{t-1}n_t - \overline{x})}.$$

This implies a wage rate of

$$w_t^s = (1-\alpha)A \left[\frac{s_{t-1}}{(1-\sigma)(n_{t-1} - zn_{t-1}n_t - \overline{x})}\right]^{\alpha}, \tag{50}$$

which is a function of s_{t-1}, n_{t-1}, and n_t. In addition, the following condition must hold

$$k_t > \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}.$$

Unfortunately, the new model with policy intervention cannot be solved analytically. The dynamics of all endogenous variables and their steady states must be solved numerically. The procedure is stated as follows. Given $n_0 \in (\underline{n}, \frac{1+\sqrt{1-4\overline{x}z}}{2z})$, $s_0 > 0$, and a feasible $\theta > 0$:[16]

1. Eqs. (50)(43)(46) uniquely determine n_t. Denote the solution as n_t^f. This is the fertility choice when unskilled workers spend their full time on caring activities.

2. Eqs. (49)(43)(46) uniquely determine n_t. Denote the solution as n_t^p. This is the fertility choice when unskilled workers split their time between caring and working in the final goods sector.

3. The skilled worker ratio is determined by (35), specifically

$$\phi_t = \min\left\{\left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha} \frac{s_{t-1}}{(1-\sigma)n_{t-1}}, 1 - zn_t^f - \frac{\overline{x}}{n_{t-1}}\right\}.$$

   where if

$$\phi_t = \begin{cases} 1 - zn_t^f - \frac{\overline{x}}{n_{t-1}}, \text{ then } n_t = n_t^f, w_t^s = (1-\alpha)A\left[\frac{s_{t-1}}{(1-\sigma)(n_{t-1} - zn_{t-1}n_t^f - \overline{x})}\right]^{\alpha}, \\ \left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha} \frac{s_{t-1}}{(1-\sigma)n_{t-1}}, \text{ then } n_t = n_t^p, w_t^s = Ab/(1-\sigma). \end{cases} \tag{51}$$

[16] This is because not every level of child allowance is feasible. A high allowance implies a higher tax rate, which can be destructive to the household's budget.

The equilibrium of this economy is the sequence of $\{(n_t, \phi_t, w_t^s, \tau_t, s_t, k_t)\}_{t=1}^{\infty}$ where (n_t, ϕ_t, w_t^s) are uniquely solved using (51) (following the steps above) and (τ_t, k_t, s_t) are given by (43)(48)(45), respectively.

We now present a numerical simulation to demonstrate the effectiveness of this policy. Recall that the nursing hell scenario may occur under the following parameter set: $\overline{x} = 0.8, \beta = 0.99^{120}, \gamma = 0.271, \alpha = 0.36, \sigma = 0.25, z = 0.16, A = 6, b = 0.2$. This implies $\underline{n} = 0.94$, which we have seen in the previous simulation. Without policy intervention, i.e., $\theta = 0$, fertility will converge to this value. On this path, the dual caregiving slowly eats up all resources, leaving almost nothing for consumption and savings as the fertility approaches $\underline{n}$. To steer the economy away from this outcome, we introduce a small child allowance of $\theta = 0.01$. The government implements this policy at time $t = 1$, and adjusts the tax rates at every subsequent generation to maintain the same level of child allowance. Although modest, this pronatalist policy is sufficient to prevent fertility from declining toward $\underline{n}$, while ensuring the tax burden does not exceed the household's feasible income. The dynamics are reported in Table 3.

Table 3: Dynamics when child allowance $\theta = 0.01$ is implemented from $t = 1$. The column u^0 indicates the welfare when no child allowance program is implemented.

t	τ_t	n_t	s_t	c_t	ϕ_t	k_t	u_t	$u_t^0(\theta = 0)$
0	0.00000	3.00000	0.10000					
1	0.03855	1.61565	0.12546	0.41908	0.47483	0.09360	-1.16983	- 1.1147
2	0.03235	1.34224	0.12420	0.41486	0.29008	0.35693	-1.42120	- 1.3676
3	0.03438	1.22721	0.10686	0.35694	0.20763	0.59423	-1.74425	- 1.7154
4	0.03910	1.16465	0.08916	0.29783	0.16177	0.71767	-2.05924	- 2.0627
5	0.04486	1.12577	0.07513	0.25096	0.13298	0.76764	-2.33768	- 2.3781
6	0.05093	1.09953	0.06464	0.21590	0.11345	0.78438	-2.57565	- 2.6577
7	0.05697	1.08084	0.05680	0.18974	0.09948	0.78786	-2.77722	- 2.9056
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	0.11523	1.01065	0.02626	0.08771	0.04673	0.74138	-3.95845	- 9.8360

The policy does improve the steady state fertility to 1.01, which is higher than $\underline{n} = 0.94$. More manpower leads to a lower burden of caregiving on society. The skilled worker ratio reaches 4.67% from a near-zero level, allowing for more consumption and savings. Although the economy is no longer in a *nursing hell* path, maintaining the child allowance ultimately requires a high wage income tax. In our case, an allowance of 0.01 per child necessitates a steady-state wage income tax of 11.52%.

As argued at the beginning of this section, there exists an intergenerational externality: a larger number of children lowers the per-child burden of supporting aging parents (and vice versa). However, parents do not take such effects into account when making fertility decisions, as they do not consider the welfare of their children. The child allowance policy addresses this inefficiency by subsidizing child-rearing, raising fertility so that the economy reaches a more efficient allocation. An important caveat is that this policy does not necessarily bring forth a Pareto improvement. A comparison of u_t in Table 3 with the no-policy benchmark in Table 1 shows that welfare declines for the first three generations after the policy implementation. This decline occurs because the policy redirects part of household resources from consumption to child-rearing. Under our parameterization, the fertility gains are insufficient to offset the consumption losses, resulting in lower welfare for some generations. From the fourth generation onward, however, welfare improves substantially. By encouraging higher fertility, the policy sustains a sufficient supply of skilled labor, which in turn raises output and supports higher consumption levels.

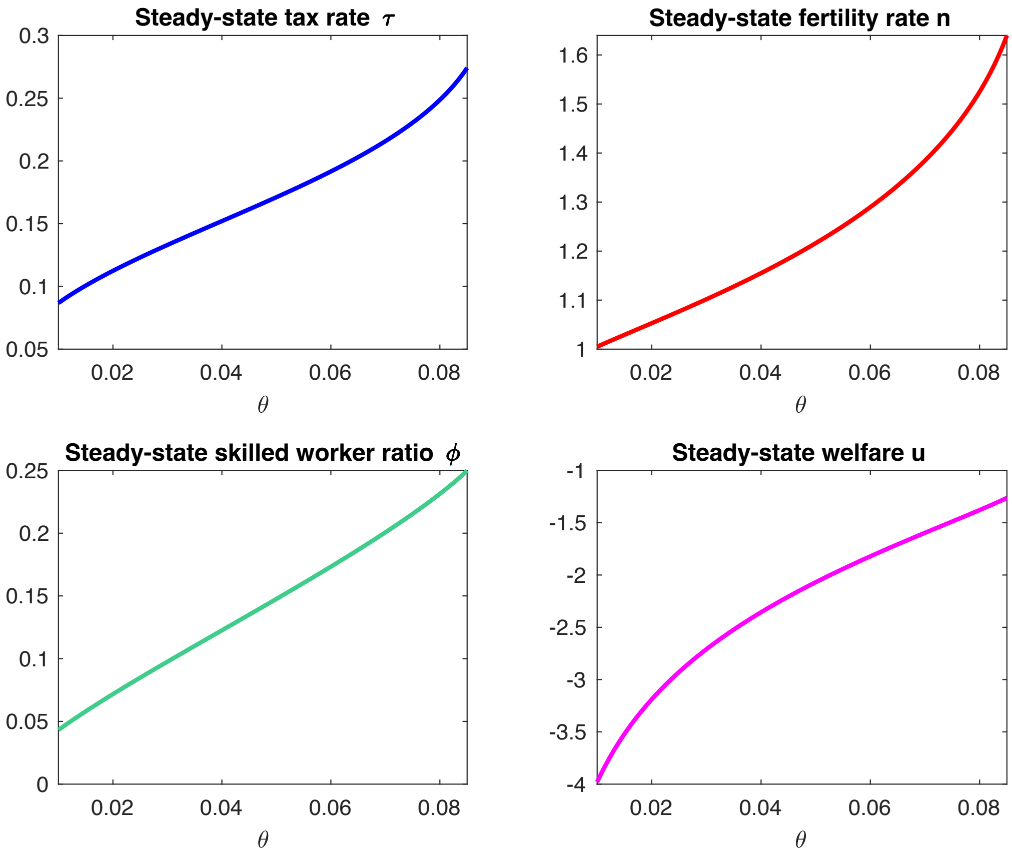


Figure 5: Steady-state outcomes with different values of child allowances.

We now examine the steady-state outcomes under different levels of child allowances. Although the fertility policy can lead to better steady-state outcomes, not every level of child allowance policy is feasible. The government cannot implement a too generous child allowance policy as doing so may require an unreasonably high tax rate, i.e., τ can exceed 1. To that end, we conduct a numerical evaluation over the policy range $\theta \in [0.01, 0.1]$ and investigate the corresponding steady-state outcomes.

Fig. 5 confirms that when $\overline{x}z \in [\eta, 1/4]$, a higher child allowance leads to higher fertility in the steady state, and therefore, a higher level of welfare. Such an effect is expected. A higher child allowance means society cares more about the future generations, which implies a higher δ, resulting in a higher θ. Thanks to the increase in the population size, the reduced dual caregiving burden allows agents to supply more skilled labor, thereby increasing the net income per middle-aged agent. It is also worth noting that the maximum feasible θ that can be implemented under this parameterization is 0.085, which requires a tax rate of 27%. Any child allowance larger than this threshold would be impossible. If society cares a lot about future generations in the steady state, the maximum feasible tax rate should be implemented to achieve the highest level of welfare in the steady state.

7 What happens if $\overline{x}z > 1/4$?

So far, the possibility that $\overline{x}z > 1/4$ has been excluded by assumption. We are now in a position to reveal the reason for this. In a nutshell, such an economy will inevitably fail to operate.

Let's try the following thought experiment. Suppose that the middle-aged generation in period t invests all of their endowed labor in child and elderly care, then the following equation holds:

$$1 - zn_t - \overline{x}/n_{t-1} = 0$$

or

$$n_t = (1 - \overline{x}/n_{t-1})/z. \tag{52}$$

When $\overline{x}z > 1/4$, on a plane with n_{t-1} on the horizontal axis and n_t on the vertical axis, the graph for (52) lies below the 45-degree line, i.e.

$$n_t = (1 - \overline{x}/n_{t-1})/z < n_{t-1},$$

meaning that starting from any initial value, the fertility rate takes on a negative value after a finite time (see Figure 6). In other words, for some periods after the beginning of the economy, all of the labor imparted to the middle-aged generation combined will be insufficient to care for their parents' generation.

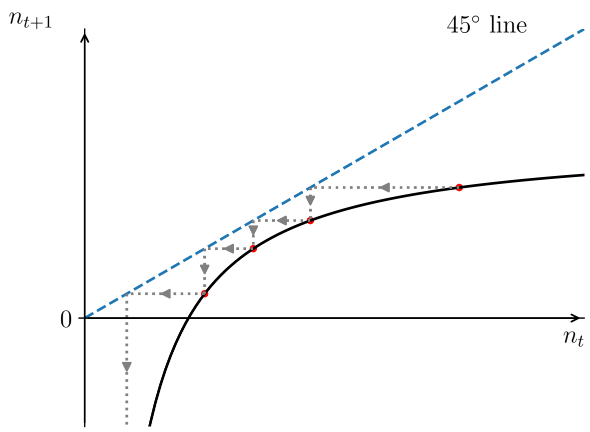


Figure 6: Dynamics of n_t when $\overline{x}z > 1/4$.

This result asserts that when $\overline{x}z > 1/4$, the manpower needed to care for children and the elderly will exceed the labor supply of the middle-aged generation in finite periods. Even if all labor is invested in caregiving, there will be a generation in a few periods that will not be able to care for their own parents' generation. If some portion of the labor force were invested in productive activities, such a generation would emerge much earlier.

This failure cannot be avoided by the policy examined in the previous section. That policy was to raise the birth rate of the middle-aged generation, thereby preventing the economy from descending into a *nursing hell*. However, the above proposition implies that the maximum feasible birth rate is already chosen in each period, leaving no room for the policy to be effective. In the first place, that policy was effective because the following equation holds for n slightly larger than $\underline{n}$:

$$1 - zn - \frac{\overline{x}}{n} > 0,$$

which is possible only when $\overline{x}z \leq 1/4$. Thus, if the burden of care beyond a certain critical level, it breaks down this condition and threatens the very survival of the economy.

If anything can rescue the economy from this predicament, it is the type of public care system analyzed by Yakita (2023). When public care proves more efficient than family-based care, it can lower the effective value of $\overline{x}$, thereby restoring the condition $\overline{x}z < 1/4$ required for sustainable growth. The widespread reports of "granny dumping" around the world underscore the urgent need for such public provision. In this sense, the present analysis and Yakita (2023) are highly complementary.[17]

8 Conclusion

This paper has examined the impact of the dual caregiving burden on economic growth in the framework of an overlapping generations model where agents are siblings to one another. The model concludes that a heavy dual care burden can negatively impact economic growth. Specifically, a sufficiently large dual care burden often leads to extremely low fertility choices, which impose negative externalities on subsequent generations. These results help explain the rationale for pronatalist policies currently adopted in many countries to address declining fertility. More importantly, once the care burden exceeds a certain critical level, even the survival of the economy is at stake. Given the growing time demands of both childcare and eldercare, the implications of this analysis are highly relevant. The dual care burden must therefore be maintained at a sufficiently low level to achieve sustainable growth.

These conclusions raise the following natural question: why has human society survived to the present day, although the survival condition identified in this paper seems too restrictive to be met? One plausible answer would be that humans have had a short life span for a long time, and not many people could reach old age. This would have contributed greatly to keeping the burden of elder care at a low level. Today, however, these circumstances have drastically changed, as Wilmoth (2011) wrote:

> Life expectancy has been increasing not only in industrialized societies but also around the world. According to estimates by the United Nations, life expectancy at birth for the world as a whole has risen from around 46 years in 1950 to approximately 68 years in 2009. During this same time interval, life expectancy at birth has increased from 65 to 77 years for the more developed regions and from 40 to 66 years for the less developed regions. Even the least developed countries have experienced a rise in life expectancy at birth over this period, from 35 to 57 years. (pp.156-158)

[17] A modified version of our model that incorporates a state-funded LTC similar to Yakita (2023) is shown in Appendix E. Indeed, when the public care producitivty is higher than family care, and the government is willing to support at least a portion of the overall elderly care, the economy can function properly and admit a positive steady-state fertility even if $\overline{x}z > 1/4$.

This global increase in longevity is the fruit of scientific progress since the Industrial Revolution. Yet while science has fulfilled humanity's desire to live longer, it has also created the new challenge of dual caregiving.

To highlight the critical role of the care burden while keeping the model simple, the present analysis abstracts from modeling longevity explicitly. Incorporating this dimension would be a natural and important extension, particularly for studying demographic transitions. Relevant contributions include Chakraborty (2004); Futagami and Konishi (2019); Tran (2022); Chen et al. (2024). Addressing this issue, however, lies beyond the scope of the current paper and is left for future research.

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APPENDIX

A Proof of Proposition 1

Proof. Equation (27) implies that its dynamical system has two stationary points, $\underline{n}$ and $\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}$. The relationship between them depends on the value of $\overline{x}z$. Specifically, when $\overline{x}z \in (0, \eta)$,

$$\underline{n} < \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}. \tag{53}$$

When $\overline{x}z = \eta$,

$$\underline{n} = \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}. \tag{54}$$

When $\overline{x}z \in (\eta, 1/4]$,

$$\underline{n} > \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}. \tag{55}$$

To see this fact, note that comparing $\underline{n}$ and $\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}$ is the same as comparing $\frac{(z\underline{n})^2}{\overline{x}z}$ and $\frac{\gamma}{1+\gamma+\beta}$. For example, (53) is equivalent to the following inequality:

$$\frac{(z\underline{n})^2}{\overline{x}z} < \frac{\gamma}{1+\gamma+\beta}. \tag{56}$$

Using (7), we can rewrite the LHS of (56) as

$$\frac{(z\underline{n})^2}{\overline{x}z} = \frac{(1-\sqrt{1-4\overline{x}z})^2}{4\overline{x}z}.$$

By differentiating it with $\overline{x}z$, we have

$$\frac{d}{d(\overline{x}z)}\frac{(z\underline{n})^2}{\overline{x}z} = \frac{1-\sqrt{1-4\overline{x}z}}{4(\overline{x}z)^2}\left(\frac{1}{\sqrt{1-4\overline{x}z}} + \sqrt{1-4\overline{x}z} - 1\right).$$

This derivative is positive since the arithmetic-geometric mean relationship implies that

$$\frac{1}{\sqrt{1-4\overline{x}z}} + \sqrt{1-4\overline{x}z} - 1 \geq 2\left(\frac{1}{\sqrt{1-4\overline{x}z}} \cdot \sqrt{1-4\overline{x}z}\right)^{1/2} - 1 = 1 > 0.$$

Thus, we can state that $(z\underline{n})^2/\overline{x}z$ is an increasing function of $\overline{x}z$. In addition,[18]

$$\lim_{\overline{x}z \to 0} \frac{(z\underline{n})^2}{\overline{x}z} = 0, \qquad \text{and} \qquad \lim_{\overline{x}z \to 1/4} \frac{(z\underline{n})^2}{\overline{x}z} = 1.$$

[18] To derive the first equation, use L'Hôpital's theorem.

These results jointly mean that there is a unique value of $\overline{x}z$ such that

$$\frac{(z\underline{n})^2}{\overline{x}z} = \frac{(1-\sqrt{1-4\overline{x}z})^2}{4\overline{x}z} = \frac{\gamma}{1+\gamma+\beta}.$$

By solving this equation for $\overline{x}z$, we can obtain the value of η. Since $(z\underline{n})^2/\overline{x}z$ is incresing with $\overline{x}z$, when $\overline{x}z < \eta$, the following inequality is true:

$$\frac{(z\underline{n})^2}{\overline{x}z} < \frac{\gamma}{1+\gamma+\beta},$$

which is equivalent to (53). Likewise, when $\overline{x}z > \eta$, the following is true:

$$\frac{(z\underline{n})^2}{\overline{x}z} > \frac{\gamma}{1+\gamma+\beta},$$

which is equivalent to (55). All that is needed to obtain the desired result is to check the positional relationship between the graph of (27) and the 45-degree line in the coordinate plane with n_t on the horizontal axis and n_{t+1} on the vertical axis. To accomplish this task, let us rewrite (27) as follows:

$$n_t - n_{t-1} = -\frac{n_{t-1}-\underline{n}}{n_{t-1}}\left[n_{t-1} - \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}\right]. \tag{57}$$

When the RHS of (57) is positive, the graph of (27) is above the 45-degree line; when it is negative, it is below. When $\overline{x}z \in (0,\eta)$, the RHS of (57) is positive for $n_{t-1} \in \left(\underline{n}, \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}\right)$ and negative for $n_{t-1} \in \left(\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}, +\infty\right)$, which means that n_t is approaching $\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}$, independent of its initial value, as shown in Figure 3a. On the other hand, when $\overline{x}z \in [\eta, 1/4]$, the RHS of (57) is negative for $n_{t-1} \in (\underline{n}, +\infty)$, which means that starting from any point larger than $\underline{n}$, n_t is decreasing to $\underline{n}$ over time, as shown in Figure 3b. □

B Proof of Proposition 2

Proof. The fertility decision implies

$$\begin{aligned}
n_t - \underline{n} &= \frac{\gamma}{1+\gamma+\beta}\frac{(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)+\theta\underline{n}}{(1-\tau_t)(1-\sigma)w_t^s z-\theta} \\
&> \frac{\gamma}{1+\gamma+\beta}\frac{(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)}{(1-\tau_t)(1-\sigma)w_t^s z-\theta} \\
&> \frac{\gamma}{1+\gamma+\beta}\frac{(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)}{(1-\tau_t)(1-\sigma)w_t^s z} \\
&= \frac{\gamma}{(1+\gamma+\beta)z}\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right).
\end{aligned}$$

Assume that $n_0 > \underline{n}$. The limits when n_{t-1} is closed to $\underline{n}$ is

$$\lim_{n_{t-1}\to\underline{n}^+}(n_t-\underline{n}) > \lim_{n_{t-1}\to\underline{n}^+}\frac{\gamma}{(1+\gamma+\beta)z}\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right) = 0$$

since $1 - z\underline{n} - \overline{x}/\underline{n} = 0$ by (7). In other words, $n_t > \underline{n}\ \forall t$ given $n_0 > \underline{n}$. □

C Note on the division of labor

When the economy follows the *sustainable growth* path, characterized by $\overline{x}z < 1/4$, we can determine the steady-state division of labor – specifically, whether unskilled workers allocate some of their labor to final goods production or dedicate all of it to caregiving. In the former case, they function as part-time caregivers, while in the latter, they become full-time caregivers. Regardless of either outcome, the steady-state fertility n^* is

$$n^* = \frac{\gamma \overline{x}}{(1+\beta+\gamma)z\underline{n}}.$$

Assume that the steady state ϕ^* is to have unskilled workers specialized in the dual caregiving duty, it is characterized by the following steady-state equations

$$\begin{cases} n^* &= \dfrac{\gamma \overline{x}}{(1+\beta+\gamma)z\underline{n}}, \\ s^* &= \dfrac{(1-\sigma)(1-\alpha)\beta A}{1+\beta+\gamma}(k^*)^{\alpha}(1-z\underline{n}-\overline{x}/n^*), \\ \phi^* &= 1 - zn^* - \overline{x}/n^*, \\ k^* &= \dfrac{s^*}{(1-\sigma)\phi^* n^*}. \end{cases}$$

Combining all these equations yields

$$k^* = \left[\frac{\beta(1-\alpha)A}{1+\beta+\gamma}\cdot\chi(n^*)\right]^{\frac{1}{1-\alpha}}. \tag{58}$$

where $\chi(n^*)$ is given by:

$$\chi(n^*) = \frac{1 - z\underline{n} - \overline{x}/n^*}{(1 - zn^* - \overline{x}/n^*)n^*},$$

Since the inequality (32) must hold in this equilibrium, equations (32) and (58) jointly imply that

$$\left[\frac{\beta(1-\alpha)A}{1+\beta+\gamma}\cdot\chi(n^*)\right]^{\frac{1}{1-\alpha}} > \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}$$

or

$$A > \frac{1+\beta+\gamma}{(1-\alpha)\beta\chi(n^*)}\left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{\frac{1-\alpha}{\alpha}}.$$

This result shows that when labor productivity is sufficiently large, unskilled workers should spend all their time on caregiving duties. Otherwise, when labor productivity is sufficiently low, unskilled workers work partially on the market while simultaneously fulfilling the dual care responsibility at home.

D Constructing the phase diagram

D.1 Optimization

Let $Z_t \equiv (1-\sigma)w_t^s\left(1 - zn_t - \frac{\overline{x}}{n_{t-1}}\right)$, the FOCs with λ_t as the Lagrangian multiplier are

$$
\begin{aligned}
(c_t): \quad & \lambda_t = 1/c_t, \\
(n_t): \quad & \frac{\gamma}{n_t - \underline{n}} + \lambda_t \frac{\partial Z_t}{\partial n_t} + \delta \lambda_{t+1} \frac{\partial Z_{t+1}}{\partial n_t} = 0.
\end{aligned}
$$

The last equation can be computed as

$$
\frac{\gamma}{n_t - \underline{n}} - \lambda_t (1-\sigma) w_t^s z + \delta \lambda_{t+1} (1-\sigma) w_{t+1}^s \frac{\overline{x}}{n_t^2} = 0. \tag{59}
$$

Since $\lambda_t = 1/c_t$, and $\lambda_{t+1} = 1/c_{t+1}$, and $d_{t+1} = \beta R_{t+1} c_t$, we can use the budget constraint to solve for c_t:

$$
c_t = \frac{(1-\sigma)w_t^s(1 - zn_t - \overline{x}/n_{t-1})}{1+\beta}.
$$

Substituting into (59) to obtain:

$$
\frac{\gamma}{n_t - \underline{n}} + \delta \frac{(1+\beta)}{1 - zn_{t+1} - \frac{\overline{x}}{n_t}} \cdot \frac{\overline{x}}{n_t^2} = \frac{z(1+\beta)}{1 - zn_t - \frac{\overline{x}}{n_{t-1}}}. \tag{60}
$$

D.2 Constructing the phase diagram

The system contains

$$
l_{t+1} = \Psi(n_t, l_t) = \frac{\delta(1+\beta)\overline{x}(n_t - \underline{n})l_t}{n_t^2[(1+\beta)z(n_t - \underline{n}) - \gamma l_t]} \tag{61}
$$

and

$$
n_{t+1} = \Phi(n_t, l_t) = \frac{1}{z}\left[1 - \frac{\overline{x}}{n_t} - \frac{\delta(1+\beta)\overline{x}(n_t - \underline{n})l_t}{n_t^2[(1+\beta)z(n_t - \underline{n}) - \gamma l_t]}\right]. \tag{62}
$$

The system has two steady states $(\underline{n}, 0)$ and $(n^*, 1 - zn^* - \overline{x}/n^*)$. Note that when $\overline{x}z \leq \eta$, then n^* is solved following Appendix C.

First, in (61), let $l_{t+1} = l_t = \ell \neq 0$, solving for ℓ:

$$
\begin{aligned}
1 &= \frac{\delta(1+\beta)\overline{x}(n_t - \underline{n})}{n_t^2[(1+\beta)z(n_t - \underline{n}) - \gamma \ell]} \\
(1+\beta)z(n_t - \underline{n}) - \gamma \ell &= \frac{\delta(1+\beta)\overline{x}(n_t - \underline{n})}{n_t^2} \\
\ell &= \frac{(n_t - \underline{n})(1+\beta)}{\gamma}\left(z - \frac{\delta \overline{x}}{n_t^2}\right).
\end{aligned} \tag{63}
$$

This nullcline exists only when $n_t > \sqrt{\delta\overline{x}/z}$. To see the shape of the nullcline, differentiate ℓ with respect to n_t:

$$\begin{aligned}\frac{d\ell}{dn_t} &= \frac{(1+\beta)}{\gamma}\left(z - \frac{\delta\overline{x}}{n_t^2}\right) + \frac{1+\beta}{\gamma}\cdot\frac{2\delta\overline{x}}{n_t^3}(n_t - \underline{n}) \\ &= \frac{1+\beta}{\gamma}\left(z + \frac{\delta\overline{x}}{n_t^2} - \frac{2\delta\overline{x}\underline{n}}{n_t^3}\right).\end{aligned}$$

The sign of $d\ell/dn_t$ depends on the sign of the following cubic function:

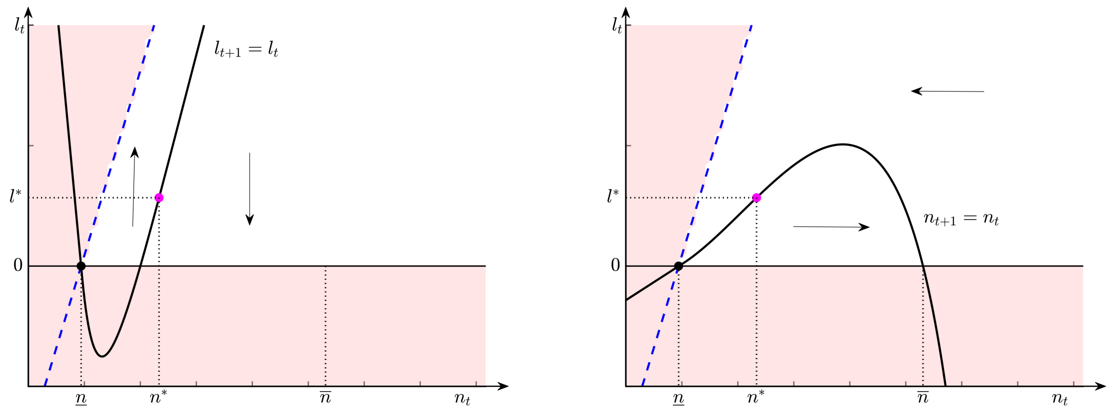
$$\xi(n_t) \equiv zn_t^3 + \delta\overline{x}n_t - 2\delta\overline{x}\underline{n}.$$

Let $\hat{n}_1$ be the real root of $\xi(n_t) = 0$.[19] Assuming the discriminant and coefficients are such that $\hat{n}_1$ is the unique real root and that $\dfrac{d\ell}{dn_t}$ has the same sign as $\xi(n_t)$, we have $\dfrac{d\ell}{dn_t} < 0$ for $n_t < \hat{n}_1$ and $\dfrac{d\ell}{dn_t} > 0$ for $n_t > \hat{n}_1$.

To see the trajectory of when there is a change in l_t, note that

$$\frac{\partial\Psi(n_t, l_t)}{\partial n_t} = \frac{(n_t - \underline{n})[-2(1+\beta)zn_t^2 - 2(1+\beta)z\underline{n}n - 2\gamma l_t n_t] - \gamma l_t n_t}{\{n_t^2[(1+\beta)z(n_t - \underline{n}) - \gamma l_t]\}^2} < 0 \text{ for } n_t > \underline{n},$$

so the points on the right the nullcline have $l_{t+1} < l_t$, and the points on the left have $l_{t+1} > l_t$, as demonstrated in panel 7a of Fig. 7.



(a) The $l_{t+1} = l_t$ nullcline.

(b) The $n_{t+1} = n_t$ nullcline.

Figure 7: Construction of the phase plane.

For the other part of the problem, we let $n_{t+1} = n_t = n$ in (62) and solve for l_t to obtain the n-nullcline:

$$l_t = 1 - \frac{\overline{x}}{n} - zn. \tag{64}$$

[19] If one depresses the cubic to $u^3 + pu + q = 0$ then, when $\Delta = (q/2)^2 + (p/3)^3 \geq 0$, the real root is given by Cardano's formula

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}},$$

where $p = \delta\overline{x}/z$ and $q = -2\delta\overline{x}\underline{n}/z$. We can recover $\hat{n}_1$ from u by subsitution.

For each value of n, we obtain a corresponding value for l. The critical point is $\hat{n}_2 = \sqrt{\overline{x}/z}$. The nullcline is increasing in n_t when $n_t < \hat{n}_2$ and decreasing when $n_t > \hat{n}_2$. To see the trajectory of when there is a change in n_t, note that

$$\frac{\partial \Phi(n_t, l_t)}{\partial l_t} = -\frac{\delta(1+\beta)^2 \overline{x}(n_t - \underline{n})^2}{n_t^2[(1+\beta)z(n_t - \underline{n}) - \gamma l_t]^2} < 0,$$

so the points above the nullcline have $n_{t+1} < n_t$ and the points to the left have $n_{t+1} > n_t$, as indicated in panel 7b of Fig. 7. Note that there are two values of n_t that results in $l_t = 0$, one is $\underline{n}$ as we are inspecting, the other is $\overline{n} = \frac{1+\sqrt{1-4\overline{x}z}}{2z}$, i.e., the larger root of Eq.(6), but it is not an equilibrium point.

Combining these two pieces of information, the phase plane of the system can be portrayed. Moreover, we restrict our attention to only the values of $n_t > \underline{n}$. To ensure that a positive l_{t+1} is attainable given a historical l_t and n_t, we restrict the feasible set to $0 \leq l_t \leq (1+\beta)z(n_t - \underline{n})/\gamma$. The infeasible region is the shaded area as seen in Fig. 7 (and Fig. 4).

The graphical presentation in Fig.4 shows that $(\underline{n}, 0)$ is a source and $(n^*, 1 - zn^* - \overline{x}/n^*)$ is a saddle. Since the former is a source point and the latter is a saddle point, it is almost certain that the economy will converge to the latter in the long run. Note that this condition holds only for sufficiently large altruism weight $\delta > \underline{\delta}$ when $\overline{x}z \in (\eta, 1/4)$ or when $\overline{x}z \leq \eta$. This situation is depicted in Fig. 4a.

On the other hand, when $\overline{x}z = 1/4$ or when $\overline{x}z \in (\eta, 1/4)$ with $\delta < \underline{\delta}$, the two points eventually merge at $\underline{n}$. Equation (63) governing the $l_{t+1} = l_t$ nullcline implies that $d\ell/d\delta < 0$, meaning that when δ decreases, ℓ increases. Furthermore, the existence condition $n_t > \sqrt{\delta\overline{x}/z}$ approaches zero when $\delta \to 0$, implying that the nullcline $l_{t+1} = l_t$ extends further to the left as δ decreases. Since the l-nullcline shifts upwards and extends to the left as δ decreases, the steady state $(n^*, 1 - zn^* - \overline{x}/n^*)$ moves toward lower l and lower n. At the critical value $\delta = \underline{\delta}$, the two steady states merge into one. For $0 \leq \delta < \underline{\delta}$, the steady state $(n^*, 1 - zn^* - \overline{x}/n^*)$ vanishes. Only the $(\underline{n}, 0)$ steady state remains and becomes the only saddle point. This situation is depicted in Fig. 4b.

E A simple model with state-funded LTC

Assume now that the government also provides some LTC services, funded by a lump-sum tax similar to Yakita (2023). The government supply of LTC is h^G, which implies that the family elderly care supply is $\overline{x} - h^G$ ($h^G \leq \overline{x}$).

We assume that public care is produced by careworkers who are paid the unskilled wage, but their productivity is $\mu > 1$, meaning one unit of time of a public careworker provides μ units of care.[20] Therefore, the government needs to hire the care labor L_t^G such that

$$\mu L_t^G = h^G N_{t-1}.$$

For careworkers to be indifferent between working for the state or the market, they are paid the same wage rate of w_t^u. A lump-sum tax T_t is levied on each individual to

[20]In a normal family-based set up, one unit of time provides one unit of care, but in here, under the LTC program, one unit of time can provide $\mu > 1$ units of care. The higher productivity of the public LTC implies there may be some benefits from economies of scale and better management.

finance the LTC program. The government budget balancing equation is

$$T_t N_t = w_t^u L_t^G = w_t^u \frac{h^G N_{t-1}}{\mu}$$

or

$$T_t = w_t^u \frac{h^G}{\mu n_{t-1}}.$$

Now, the household's budget constraint, in per-member terms, must be modified to include the tax.

$$c_t + \frac{d_{t+1}}{R_{t+1}} + T_t = (1-\sigma) w_t^s \left(1 - z n_t - \frac{\overline{x} - h^G}{n_{t-1}}\right).$$

By substituting the tax

$$c_t + \frac{d_{t+1}}{R_{t+1}} = (1-\sigma) w_t^s \left(1 - z n_t - \frac{\overline{x} - h^G}{n_{t-1}}\right) - w_t^u \frac{h^G N_{t-1}}{\mu n_{t-1}}.$$

Using the equilibrium condition $(1-\sigma) w_t^s = w_t^u$, then we arrive at

$$\begin{aligned} c_t + \frac{d_{t+1}}{R_{t+1}} &= (1-\sigma) w_t^s \left[1 - z n_t - \frac{\overline{x}}{n_{t-1}} + \left(1 - \frac{1}{\mu}\right) \frac{h^G}{n_{t-1}}\right] \\ &= (1-\sigma) w_t^s \left(1 - z n_t - \frac{\overline{x} - h^G(1-1/\mu)}{n_{t-1}}\right). \end{aligned}$$

We can use the same solutions as in (22)–(24). The fertility dynamics are

$$n_t - \underline{n} = \frac{\gamma}{(1+\gamma+\beta) z} \left(1 - z n_t - \frac{\overline{x} - h^G(1-1/\mu)}{n_{t-1}}\right). \tag{65}$$

The critical condition for the economy to avoid collapse (i.e., for there to be a positive steady-state fertility) is that the effective care burden is not too high. Specifically, such a condition is

$$\overline{x} - h^G(1-1/\mu) \leq 1/4z. \tag{66}$$

Thus, even if $\overline{x} z > 1/4$, by supplying a sufficiently large state-funded LTC services $h^G > (\overline{x} - 1/4z)/(1-1/\mu)$, the dire situation can be avoided. It is sufficient to show that a positive steady state fertility exists given when $\overline{x} z > 1/4$, given the existence of a state-funded LTC under care productivity μ.

Define effective family care as

$$B \equiv \overline{x} - h^G(1-1/\mu).$$

Assume that a steady state fertility, denoted by n^{**} exists, then (65) becomes

$$n^{**} - \underline{n} = \frac{\gamma}{(1+\gamma+\beta) z} \left(1 - z n^{**} - \frac{B}{n^{**}}\right)$$

For a positive steady state to exist, the following equation must hold

$$1 - zn^{**} - \frac{B}{n^{**}} \geq 0$$

or

$$B \leq 1/4z.$$

This implies

$$h^G \geq \frac{\overline{x} - 1/4z}{1 - 1/\mu}.$$

In other words, so long as $\mu > 1$ and the government can supply $h^G \in \left[\frac{\overline{x} - 1/4z}{1 - 1/\mu}, \overline{x}\right]$, then the economy admits a positive steady state fertility even if $\overline{x}z > 1/4$.