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Dual Caregiving and Overlapping Generations

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Abstract

This paper addresses the issue of dual caregiving duties (child-rearing and elderly care) and how they can affect the stability of an economy. In this model, agents are considered siblings to one another, making collective decisions on fertility, savings, and division of labor to maximize their welfare. In the long run, the dual care burden plays a significant role. If it is sufficiently small, the economy is on a sustainable growth path. If it is sufficiently large, the economy can be locked in a nursing hell path where the fertility is so low that almost all resources must be dedicated to caregiving, leaving little for income generation. In this case, the government can provide child allowances to incentivize childbearing and mitigate the problem. More critically, if the caregiving burden surpasses a certain threshold, the economy may face a structural collapse. In this case, even a pronatalist policy is infeasible to implement as raising more children will exceed the economy's capacity.

Keywords: dual caregiving, endogenous fertility, overlapping generations

JEL Classification: E13, J13, J14, J22, J24, O11

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1 Introduction

In many advanced economies today, middle-aged adults serve as the primary caregivers in families, providing informal care for elderly parents while dedicating time to child-rearing. [1] Given the limited time available for child-rearing and participation in the labor market, individuals may determine that it is more optimal to have fewer children. However, this choice results in the next generation having a smaller sibling size, which, in turn, increases the caregiving responsibility placed on each adult.

This paper explores the implications of the dual caregiving burden faced by middle-aged adults who must care for their elderly parents and their own children at the same time. To do so, we utilize an overlapping generations framework à la Diamond (1965). The economy is populated by groups of siblings who optimize their time allocation among market work, child-rearing, and elderly care. This family structure allows us to investigate how household size – measured by the number of siblings – influences both the caregiving burden and fertility decisions, particularly when child-rearing and elderly care are time-intensive and non-transferable activities. [2]

In our model, fertility is determined endogenously under a "joy-of-giving" (or warm glow) altruism where parents value the number of children, similar to Galor and Weil (2000); Galor (2005), alongside various other studies. [3] Another crucial assumption in this paper is that children bear the responsibility of caring for dependent parents. This approach is similar to Pestieau and Sato (2008). Since this form of informal care does not necessarily generate utility, it implies that the motive for children to provide assistance to their parents can be attributed to family or cultural norms (Canta et al., 2016). Although the literature indicates three primary motives for providing informal care (pure altruism, financial ex-

[1] Related studies include those focusing on the United States (Cecchini, 2018), Japan (Niimi, 2016), South Korea (Song, 2014), and Europe (Ciani, 2012). For a global view on adult children caring for elderly parents, see Lehnert et al. (2019).

[2] For a model where elderly care is considered a consumption good, see Hashimoto and Tabata (2010). For the class of models allowing for the transfers of time across generations (in the form of grandparenting), we refer to Cardia and Ng (2003); Cardia and Michel (2004); Mizushima (2009).

[3] This approach is found in several papers concerning the implications of endogenous fertility (Hirazawa and Yakita, 2017; Futagami and Konishi, 2019). While it is different from Barro and Becker (1989)'s dynastic altruism in which parents also care about the children's utility, we adopt this formulation because of its tractability and empirical significance (de La Croix and Doepke, 2003; de la Croix and Licandro, 2013; Doepke et al., 2023).

change, and family norms), this paper focuses on the implications of family norms. Klimaviciute et al. (2017), using the data from the European Survey of Health, Ageing, and Retirement (SHARE), show that while all three have significant contributions to informal caregiving, family norms are more prevalent in Eastern and Southern Europe, while moderate altruism is more common in other regions. This is true when informal care provided by children remains a reliable, cost-effective, and accessible solution with a higher sense of satisfaction derived from kinship bonds (Bonsang, 2009) compared to the formal alternatives. Complementing this, Leroux and Pestieau (2014) shows that if children's assistance were certain, agents would rely exclusively on the family for care. Thus, aside from child-rearing, elderly care is also an important constraint for middle-aged agents.

Before reviewing relevant theoretical research related to ours, we provide some overview of empirical research on dual caregivers. Suh (2016) reports that nearly half (47%) of Americans aged 47 to 59 in 2012 had a parent aged 65 or older and were also either raising at least one child under 18 or providing financial support to an adult child aged 18 or older. The temporal burden of caregiving ranged from 11.2 to 60 hours per week, clustering at around 20 hours per week for most cases. This can have severe impacts on job security. In Japan, Kawabe et al. (2024) show that in 2019, about 53% of those who are working also provide informal dual caregiving. A significant portion of time must be spent on these duties. As indicated in Kolpashnikova and Kan (2021), caregivers in Japan (2006) typically spend 9.5 hours per day on caring, which is a longer commitment than a full-time job. A similar pattern is also found in Europe. Hoffmann and Rodrigues (2010) report that family caregivers across the EU provided 80% of all care, and 40% of informal caregivers were engaged in paid work in 2001. Given the increasing trend of old age dependency ratio and reduced fertility, these numbers will likely rise further. The direct consequences can be observed in the labor market and have been empirically verified in many studies. For instance, a higher caregiving burden can lead to displacement from traditional full-time jobs with demanding schedules (Bolin et al., 2008; Van Houtven et al., 2013; Ikeda, 2017), lower labor market participation (Leigh, 2010), more susceptible to mental health issues (Labbas and Stanfors, 2023), and the ability of workers to accumulate human capital (Skira, 2015). Given the significance of the issue, it is important to consider the long-term implications of dual caregiving on the labor market, capital accumulation and macroeconomic stability. In this regard, a theoretical analysis, as presented in this paper, can be deemed more appropriate. In particular, we examine the conditions under which the economy remains sustainable and those under which it does not, as well as identify effective policy options when the economy faces instability.

Regarding previous theoretical works, the idea of dual caregiving has been investigated in Raut and Srinivasan (1994); Chakrabarti (1999) where individuals give a constant fraction of their income to their parents in the form of old age support. Therefore, the size of the population does not matter. Different from that approach, we assume that the size of the population is important in determining the elderly care burden per person, similar to Yakita (2023). If fertility is high, the elderly care burden per sibling can be reduced as it is shared among a larger number of siblings. However, Yakita (2023) considers only the elderly care services without child-rearing services. In this paper, we assume that both elderly care and childcare services exist and can be bought from the market. When this view is adopted, we can show that the setup with tradable care labor has the exact same solutions as the optimization problem of a group of siblings.

One important extension in this paper is the inclusion of occupational choices, following the works of Kimura and Yasui (2007); Chen (2010). In deciding market work, agents can choose to provide either unskilled or skilled labor, with the latter requiring a fixed time investment in training. This distinction is important in classifying the nature of care labor as a form of unskilled labor. While individuals can spend time on either caring or market labor work, only the latter is productive and can generate income. When dual caregiving duties change, the composition of workers and available time for market work can also change, which is crucial for the operationality of the economy. As a result, the model can capture a broader aggregate impact of caregiving on the allocation of skilled labor and long-term economic dynamics.

Under this framework, the model highlights three distinct long-term scenarios: *sustainable growth*, *nursing hell*, and *structural collapse*. On the sustainable growth path, the economy has sufficient labor so that agents can simultaneously work on the market while fulfilling the dual caregiving duties. On the contrary, when the economy is on the *nursing hell* path, the dual caregiving duties will eventually consume most of the available labor, leaving little for income-generating work. In this case, we consider some pronatalist policy interventions to mitigate this unintended consequence. In particular, the government can implement a child allowance to incentivize childbearing. With the increased fertility, the subsequent generations will face a lighter burden of care as they have more siblings to share. However, in extreme circumstances, if caregiving demands exceed available resources, the survival of the economy may be at serious risk due to unsustainable fertility rates. At this point, even the aforementioned policy becomes infeasible as there is no more room for additional childbearing. A structural collapse may then become inevitable.

The rest of this paper is organized as follows. Section 2 presents the model. Section 3 defines and derives the equilibrium. Section 4 examines the effects of a child allowance policy. Section 5 discusses the validity of an assumption imposed on the burden of care. Finally, section 6 concludes.

2 Model

Time is discrete and runs from 0 to infinity. The economy consists of a continuum of agents who live through three periods: young age (childhood), middle age, and old age. All decisions are made in the middle age, where time endowment is one. Each agent is a unitary representation of a couple capable of producing offspring. The production starts its operation in period 1 and continues indefinitely.

2.1 Production

In each period, a single final good is competitively produced using the following technology:

$$Y_t = A\left[K_t^{\alpha}(L_t^s)^{1-\alpha} + bL_t^u\right], \tag{1}$$

where Y_t, K_t, L_t^s and L_t^u, respectively, denote the output of the final good, the input of physical capital, that of skilled labor, and that of unskilled labor; A, b, and α are constants satisfying $A > 0, b > 0$, and $\alpha \in (0,1)$. The production technology in (1) implies that physical capital and skilled labor are more complementary than physical capital and unskilled labor. For simplicity, unskilled labor is assumed to be a perfect substitute for capital and skilled labor. [4]

Under perfect competition, the factor prices are determined as

$$w_t^s = (1-\alpha)Ak_t^{\alpha}, \tag{2}$$

$$R_t = \alpha A k_t^{\alpha-1}, \tag{3}$$

$$w_t^u \geq Ab. \tag{4}$$

[4]See Galor and Weil (1996); Kimura and Yasui (2007); Chen (2010); Day (2016) for other uses of the same technology. The empirical validity of capital-skill complementarity can be found in Duffy et al. (2004).

where $k_t \equiv K_t / L_t^s$ is the capital-skilled labor ratio. Variables w_t^s, w_t^u and R_t, respectively, denote the wage rate of skilled labor, that of unskilled labor and the gross rental rate of capital. Note that unskilled labor is hired only when the equality in (4) is established.

2.2 Overlapping generations of agents

At the beginning of each period, a new generation of three-period-lived agents is born in the economy. They are divided into groups of siblings by who their birth parent is.

Figure 1 serves as a tool to visualize the family structure. Agents born in the same generation are considered siblings to one another and make collective decisions about time spent on dual caregiving duties and labor market work. They care for the parent at the preceding node (which is 1) and the children at the succeeding node. In summary, caregivers and care receivers make up a family, and the group of siblings refers to middle-aged agents born in the same generation.

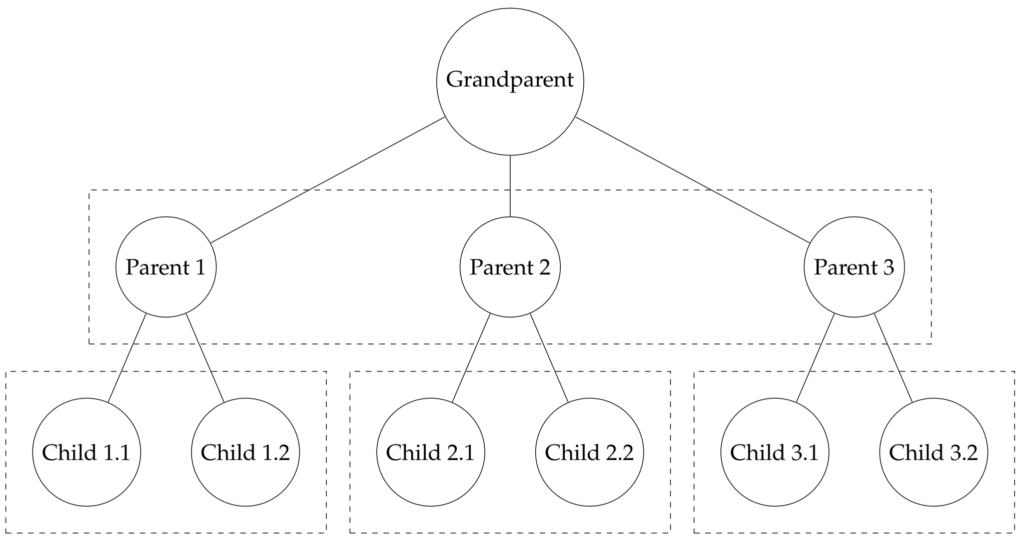


Figure 1: A visualization of a family structure with three generations.

Note: Agents enclosed by a dashed box belong to the same group and make collective decisions. For illustration purposes, each parent is assumed to have only two children, but the number can differ across generations. Although only three generations are shown here, the model features an infinite number of generations following the same structure.

In the first period of life, the agents do nothing. In the second period, each group of siblings makes collective decisions about career choice, childbearing, consumption and savings while caring for their elderly parents. In particular, child-rearing takes z units of time per child, and elderly care takes $\overline{x}$ units of time per aging parent (note that each group of siblings has only one parent). In the third period, the agents consume all the goods they saved in the previous period and die.

Consider a group of siblings who reached middle age in period t. The utility function for each middle-aged sibling is given by

$$u_t = \ln c_t + \gamma \ln(n_t - \underline{n}) + \beta \ln d_{t+1}. \quad (5)$$

where c_t, d_{t+1} and n_t represent the consumption level in period t, that in period $t+1$, and the number of children born at time t. The parameters β, γ are strictly positive, less than 1, weighing the tastes for consumption and altruism.

Note that we use a Stone-Geary preference for children. Though unconventional, this practice has found its use in Voigtlander and Voth (2013); Black et al. (2013). The value of $\underline{n}$ is strictly positive and presents a lower bound for fertility. [5] In particular, this parameter has an intuitive interpretation in our framework. Here, children care for middle-aged agents when the latter grow old. Therefore, the first few children may be born out of their parents' necessities rather than "joy". After this "subsistence" fertility level is met, the decisions about having extra children now reflect the true preferences rather than needs. Following this logic, we further define $\underline{n}$ as given by the smaller root of the following quadratic equation:

$$zn^2 - n + \overline{x} = 0, \quad (6)$$

which implies that[6]

$$\underline{n} = \frac{1 - \sqrt{1 - 4\overline{x}z}}{2z}. \quad (7)$$

If households decide to have a fertility lower than $\underline{n}$, then the following holds for any n $(> \underline{n})$:

$$1 - zn - \overline{x}/n < 0.$$

[5] In a larger extent, Baudin et al. (2015) introduce similar innovations in modeling fertility preference to allow for childlessness by setting $\underline{n}$ to negative.

[6] This assumption is also convenient for analysis. The necessary condition for the model in this paper to have a steady state is

$$\underline{n} < (1 + \sqrt{1 - 4\overline{x}z})/2z,$$

which is obviously satisfied by this assumption.

As later shown, the left-hand side presents the effective labor time. This means that the total amount of labor allocated to the middle-aged generation will not be enough to meet the long-term needs of child and elder care. Following (7), we assume that $\overline{x}$ and z satisfy:

$$\overline{x}z \leq 1/4.$$

The critical implications of $\underline{n}$ will be discussed in section 5 when we relax this assumption.

In making the collective decision stated above, this group of siblings maximizes (5), implying that all members are given the same weight, and thus that c_t, d_{t+1}, and n_t are at the same level across members.

In period t, the group of siblings is endowed with one unit of time, which is spent on training, work for income, childcare, and elder care. To work, one must first become a skilled or unskilled worker. To become a skilled worker, one must spend σ $(\in (0,1))$ units of their endowed time on training. In contrast, no such training is required to become an unskilled worker.

The population of this group is determined by the decision of their parent and thus can be expressed as n_{t-1}. Let n^s_{t-1}, n^u_{t-1}, w^s_t, w^u_t, b^s_t, b^u_t, x^s_t, and x^u_t, respectively, denote populations of skilled and unskilled workers in this group, wages of skilled and unskilled workers, time spent by one skilled worker on child care, time spent by one unskilled worker on child care, time spent by one skilled worker on elder care, and time spent by one unskilled worker on elder care. Then, on the aggregate level, the following equations hold:

$$n_{t-1} = n^s_{t-1} + n^u_{t-1}, \tag{8}$$

$$b^s_t n^s_{t-1} + b^u_t n^u_{t-1} = z n_t n_{t-1}, \tag{9}$$

$$x^s_t n^s_{t-1} + x^u_t n^u_{t-1} = \overline{x}, \tag{10}$$

$$n_{t-1}(c_t + s_t) = w^s_t n^s_{t-1}(1 - \sigma - b^s_t - x^s_t) + w^u_t n^u_{t-1}(1 - b^u_t - x^u_t). \tag{11}$$

These equations describe the aggregate constraints faced by the sibling group in period t. Define the skilled worker ratio as

$$\phi_t \equiv \frac{n^s_{t-1}}{n_{t-1}} \in [0,1].$$

We can then rewrite (9)-(11) in individual terms as

$$\phi_t b^s_t + (1 - \phi_t) b^u_t = z n_t, \tag{12}$$

$$\phi_t x^s_t + (1 - \phi_t) x^u_t = \frac{\overline{x}}{n_{t-1}}, \tag{13}$$

$$c_t + s_t = w^s_t \phi_t (1 - \sigma - b^s_t - x^s_t) + w^u_t (1 - \phi_t)(1 - b^u_t - x^u_t). \tag{14}$$

When $w_t^s > w_t^u$ and $\phi_t > 0$, the care of children and elderly parents should be carried out exclusively by unskilled workers, i.e., $b_t^s = x_t^s = 0$.

To see this point, use (12) and (13) to eliminate b_t^u and x_t^u from the RHS of (14), then we have

$$c_t + s_t = w_t^s \phi_t (1-\sigma) + w_t^u \left(1 - \phi_t - z n_t - \frac{\overline{x}}{n_{t-1}}\right) - \phi_t (w_t^s - w_t^u)(b_t^s + x_t^s),$$

which implies that having skilled workers do the care work will only reduce this group's income. As will be seen, $w_t^s > w_t^u$ and $\phi_t > 0$ in equilibrium. With $b_t^s = x_t^s = 0$, the budget constraint per middle-aged agent reduces to

$$c_t + s_t = w_t^s \phi_t (1-\sigma) + w_t^u \left(1 - \phi_t - z n_t - \frac{\overline{x}}{n_{t-1}}\right). \tag{15}$$

In old age, each member withdraws all savings for goods consumption. Their consumption is given by:

$$d_{t+1} = R_{t+1} s_t. \tag{16}$$

Since skilled and unskilled workers are essential to this economy, the choice of which type of worker to become must be indifferent to the middle-aged agents. Thus, in equilibrium, w_t^s and w_t^u must satisfy

$$w_t^s (1-\sigma) = w_t^u,$$

which makes it equally attractive to be a skilled worker as it is to be an unskilled worker. We shall see the relevance of this condition in the next subsection. This equation implies that $w_t^s > w_t^u$, and simplifies (15) as

$$c_t + s_t = w_t^s (1-\sigma) \left(1 - z n_t - \frac{\overline{x}}{n_{t-1}}\right). \tag{17}$$

Assume that the group of siblings has one leader who represents the group and makes the optimal decisions. Then, his/her problem is to maximize (5) subject to (17) and (16), by optimally choosing the values of c_t, s_t, d_{t+1}, n_t, and ϕ_t.

2.3 Market equivalence

In this subsection, we show that the problem, when viewed as a collective decision-making process as presented above, is equivalent to the problem from the perspective of individual decision-making, where care labor trading is available.

In each period, the middle-aged agents still need unskilled labor to care for their own parents and to raise their children. However, instead of collective decision-making, we assume that unskilled labor can be provided by themselves or obtained from the labor market.

Individuals make decisions at two levels. First, they decide whether to become a skilled or unskilled worker. Second, they maximize utility given their respective budget constraints. In the following, we will have a look at this decision-making process for the middle-aged generation in period t.

Consider their second level of decision-making. At this point, they have already decided whether to become skilled or unskilled workers. From Eqs.(12) and (13), on the individual level, each agent has to secure $zn_t + \overline{x}/n_{t-1}$ units of unskilled labor to care for their own parents and to provide for their children. [7]

The budget constraints for skilled labor are as follows:

$$c_t + s_t + w_t^u h_t = w_t^s \left(1 - \sigma - zn_t - \frac{\overline{x}}{n_{t-1}} + h_t\right)$$

or

$$c_t + s_t = w_t^s \left(1 - \sigma - zn_t - \frac{\overline{x}}{n_{t-1}}\right) + (w_t^s - w_t^u)h_t \tag{18}$$

where h_t is the amount of unskilled labor that can be procured from the market, satisfying $h_t \in [-(1-\sigma), zn_t + \overline{x}/n_{t-1}]$. Eq.(18) implies that when $w_t^s > w_t^u$, setting the level of h_t to $zn_t + \overline{x}/n_{t-1}$ is optimal for the skilled workers. That is, the skilled workers should outsource all necessary care labor. Such a behavior reduces (18) to

$$c_t + s_t + w_t^u zn_t = w_t^s (1-\sigma) - w_t^u \frac{\overline{x}}{n_{t-1}}. \tag{19}$$

On the other hand, the budget constraints for unskilled workers are as follows:

$$c_t + s_t = w_t^u \left(1 - zn_t - \frac{\overline{x}}{n_{t-1}}\right)$$

or

$$c_t + s_t + w_t^u zn_t = w_t^u - w_t^u \frac{\overline{x}}{n_{t-1}}. \tag{20}$$

[7] This assumption can be justified if the agents are siblings to one another and share the caregiving duties equally.

Eqs.(19) and (20) imply that if $w_t^s(1-\sigma) > w_t^u$, then all middle-aged agents will choose to become skilled workers; if $w_t^s(1-\sigma) < w_t^u$, then all middle-aged people will become unskilled workers. Therefore, for there to be both skilled and unskilled workers, the following equation must hold:

$$w_t^s(1-\sigma) = w_t^u. \tag{21}$$

Since both skilled and unskilled labor are essential to this economy, we can only assume that (21) holds in equilibrium. When (21) is true, the budget constraints for skilled and unskilled workers become the same as follows:

$$c_t + s_t = w_t^s(1-\sigma)\left(1 - zn_t - \frac{\overline{x}}{n_{t-1}}\right). \tag{22}$$

The skilled and unskilled workers maximize their utility (5) subject to (22) and their old-age consumption (16). In the previous subsection, the sibling leader maximized the utility of the representative member (5) subject to (17) and (16). As (17) and (22) are essentially the same, these two optimizations are also the same. Thus, the same results are obtained whether each individual sibling optimizes or the siblings as a whole make collective decisions.

2.4 Optimization

From the argument thus far, we can write down the problem of a representative middle-aged agent in period t as follows:

$$\max_{c_t, s_t, d_{t+1}, n_t} \ln c_t + \gamma \ln(n_t - \underline{n}) + \beta \ln d_{t+1}$$

subject to

$$\begin{aligned} c_t + s_t &= w_t^s(1-\sigma)\left(1 - zn_t - \frac{\overline{x}}{n_{t-1}}\right), \\ d_{t+1} &= R_{t+1} s_t. \end{aligned}$$

Solutions are given by:

$$c_t = \frac{1}{1+\gamma+\beta} w_t^s(1-\sigma)\left(1 - z\underline{n} - \frac{\overline{x}}{n_{t-1}}\right), \tag{23}$$

$$s_t = \frac{\beta}{1+\gamma+\beta} w_t^s(1-\sigma)\left(1 - z\underline{n} - \frac{\overline{x}}{n_{t-1}}\right), \tag{24}$$

$$d_{t+1} = \frac{\beta}{1+\gamma+\beta} R_{t+1} w_t^s(1-\sigma)\left(1 - z\underline{n} - \frac{\overline{x}}{n_{t-1}}\right), \tag{25}$$

$$n_t - \underline{n} = \frac{\gamma}{(1+\gamma+\beta)z}\left(1 - z\underline{n} - \frac{\overline{x}}{n_{t-1}}\right). \tag{26}$$

Condition (26) uniquely determines the path of fertility rate (i.e., $\{n_t\}_{t=1}^{\infty}$) for its initial condition, $n_0(> \underline{n})$. We can use (6) to rewrite (26) as

$$n_t - \underline{n} = \frac{\gamma \overline{x}}{(1+\gamma+\beta)z\underline{n}} \cdot \frac{n_{t-1} - \underline{n}}{n_{t-1}}, \tag{27}$$

implying that $n_t > \underline{n}$ for any $t \geq 1$ if $n_0 > \underline{n}$.

To close the model, we assume that, in period 1, there are old agents of measure N_0, each of whom has $s_0 > 0$ units of capital and n_0 $(> \underline{n})$ children.

3 Equilibrium

Let N_t be the population of the middle-aged agents in period t. Their dynamics are given by the following equation

$$N_t = n_{t-1}N_{t-1}. \tag{28}$$

We can then express k_t as

$$k_t = \frac{s_{t-1}N_{t-1}}{(1-\sigma)\phi_t N_t} = \frac{s_{t-1}}{(1-\sigma)\phi_t n_{t-1}}, \tag{29}$$

where the second equality is obtained from (28). Given n_{t-1} and s_{t-1}, the values of k_t and ϕ_t are determined as follows.

When $(1-\sigma)w_t^s = w_t^u > Ab$, firms are not willing to hire unskilled workers. These workers are all either raising children or caring for elderly parents. This implies that

$$\phi_t = 1 - zn_t - \frac{\overline{x}}{n_{t-1}}. \tag{30}$$

Note that $(1-\sigma)w_t^s > Ab$ can be rewritten as

$$k_t > \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}. \tag{31}$$

Using (29) to eliminate k_t from this inequality, we obtain

$$\frac{s_{t-1}}{(1-\sigma)\phi_t n_{t-1}} > \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}$$

or

$$\left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha}\frac{s_{t-1}}{(1-\sigma)n_{t-1}}>\phi_t.$$

Thus, in this case, the following must hold:

$$\left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha}\frac{s_{t-1}}{(1-\sigma)n_{t-1}}>1-zn_t-\frac{\overline{x}}{n_{t-1}}\quad(=\phi_t). \tag{32}$$

When $(1-\sigma)w_t^s = w_t^u = Ab$, firms may hire some unskilled workers. Note that $(1-\sigma)w_t^s = Ab$ can be rewritten as

$$k_t=\left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}.$$

Use (29) to eliminate k_t from this equation. Then

$$\frac{s_{t-1}}{(1-\sigma)\phi_t n_{t-1}}=\left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}$$

or

$$\phi_t=\left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha}\frac{s_{t-1}}{(1-\sigma)n_{t-1}}. \tag{33}$$

From (32) and (33), we can say that the value of ϕ_t is determined as

$$\phi_t=\min\left\{\left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha}\frac{s_{t-1}}{(1-\sigma)n_{t-1}},1-zn_t-\frac{\overline{x}}{n_{t-1}}\right\}. \tag{34}$$

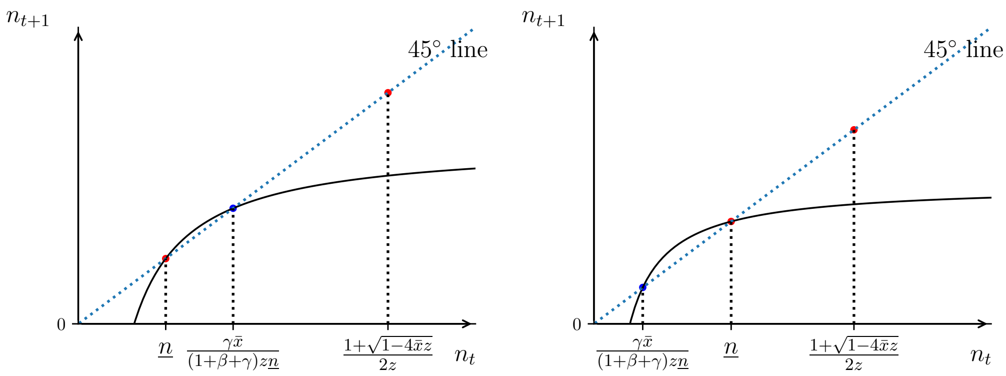
Substituting this value into (29) also yields the value of k_t.

Definition 1. The equilibrium of this economy is the sequence of endogenous variables $\{(n_t, s_t, \phi_t, k_t, w_t^s)\}_{t=1}^{\infty}$ that is uniquely determined by (2) (24) (26) (29) and (34), given $n_0 \in (\underline{n}, \frac{1+\sqrt{1-4\overline{x}z}}{2z})$ and $s_0 > 0$.

Then, we can state the following.

Proposition 1. *Define η as $\eta \equiv \gamma(1+\gamma+\beta)/(1+2\gamma+\beta)^2$. Then the equilibrium path of n_t converges to $\gamma\overline{x}/[(1+\gamma+\beta)z\underline{n}]$ if $\overline{x}z \in (0,\eta)$, and to $\underline{n}$ if $\overline{x}z \in [\eta, 1/4]$.*

Proof. See Appendix A. □



(a) When $\overline{x}z \in (0, \eta)$ (b) When $\overline{x}z \in [\eta, 1/4]$

Figure 2: The dynamics of n_t when $\overline{x}z \in (0, 1/4]$.

As visualized in Figure 2, the dynamics of fertility always converge to a higher equilibrium point. However, there is a considerable difference between what happens in equilibrium in the case of convergence to $\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}$ and in the case of convergence to $\underline{n}$. While the agents born in the distant future enjoy a decent level of consumption in the first case, their counterparts can consume almost nothing in the second case (see (23)-(26)). In the long run of the second case, almost all of the endowed labor is spent on childcare and caring for elderly parents, with little available for income-generating labor.

Numerical simulations are useful here to see this contrast. Parameters are calibrated such that $\underline{n}$ returns a positive value larger than 1 to avoid population decline. Preferences are given by $\beta = 0.99^{120} = 0.3, \gamma = 0.3$. The capital share is $\alpha = 0.36$, the training time is $\sigma = 0.4$, the elderly care time is $\overline{x} = 0.95$, and $b = 0.2$. Labor productivity is set to $A = 6$. Initial values are $s_0 = 0.1$ and $n_0 = 3$, so the economy begins with low wealth and high fertility. To generate two growth paths, we consider two economies whose only difference is the values of childcare z: one with $z = 0.1$ (so that $\overline{x}z < \eta$) and the other with $z = 0.2$ (so that $1/4 > \overline{x}z > \eta$).

Table 1 reports the equilibrium path of the economy when $\overline{x}z < \eta$. The fertility rate stabilizes at 1.6763, which is sufficiently high for agents to participate in the workforce and meet caregiving needs. Note that all unskilled labor is dedicated to caregiving duties, while skilled agents account for 26.56% of the population. [8] As everything functions properly, the economy is said to be on a *sustainable growth* path.

[8] Appendix B details the condition when this case emerges.

Table 1: Sustainable growth when $\overline{x}z < \eta$.

t	n_t	s_t	c_t	ϕ_t	k_t	u_t
0	3	0.1				
1	2.1454	0.1296	0.4329	0.3401	0.1633	-0.9471
2	1.9087	0.1222	0.4080	0.3663	0.2749	-1.1340
3	1.8058	0.1148	0.3837	0.3217	0.3315	-1.2669
4	1.7525	0.1092	0.3646	0.2986	0.3547	-1.3590
5	1.7225	0.1053	0.3518	0.2856	0.3634	-1.4204
6	1.7049	0.1028	0.3435	0.2780	0.3666	-1.4604
⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	1.6763	0.0984	0.3288	0.2656	0.3684	-1.5311

In contrast, Table 2 depicts a markedly different picture. The care burden $\overline{x}z$ now exceeds the threshold η, driving fertility towards the lower steady state of 1.27. Under these conditions, nearly all labor is allocated to caregiving, leaving the economy with an almost negligible skilled labor ratio and near-zero savings. This results in a steady and perpetual decline in welfare across generations. Eventually, almost all resources are consumed by caregiving duties. The economy is locked in a state that resembles a *nursing hell*.

Table 2: Nursing hell when $\eta \leq \overline{x}z \leq 1/4$.

t	n_t	s_t	c_t	ϕ_t	k_t	u_t
0	3	0.1				
1	1.6769	0.0962	0.3213	0.3401	0.1633	-1.7995
2	1.4426	0.0662	0.2213	0.1450	0.6596	-2.6432
3	1.3563	0.0384	0.1284	0.0702	1.0901	-3.6025
4	1.3170	0.0211	0.0706	0.0362	1.3063	-4.5913
5	1.2974	0.0115	0.0383	0.0192	1.3942	-5.5792
6	1.2872	0.0062	0.0208	0.0103	1.4273	-6.5598
⋮	⋮	⋮	⋮	⋮	⋮	⋮
59	1.2753	0.0000	0.0000	0.0000	0.7858	-21.9740

4 Effects of child allowances

In the previous section, we have seen that when $\overline{x}z \in [\eta, 1/4]$, the economy is going into the *nursing hell* where almost all of the endowed labor is spent on childcare and caring for the elderly parents, with little available for income-generating labor. As a result, the agents can consume almost nothing.

One of the causes of these catastrophes is that, in deciding how many children to have, no middle-aged agent considers how much easier the burden of caring for her with other children would be if she had one more child. Because of this negative external effect, the number of births chosen by the middle-aged generation may be below the social optimum.

This external effect could be mitigated by providing an incentive for households to give birth to more children. A natural way for the government to facilitate this is by providing a child allowance policy. In the next section, we consider a child allowance of $\theta > 0$ per birth to incentivize child-bearing. To finance this, the government imposes a tax rate of $\tau_t \in (0, 1)$ every period t on the household's income.

4.1 Environment

The household problem now becomes

$$\max_{c_t, n_t, d_{t+1}} \ln c_t + \gamma \ln(n_t - \underline{n}) + \beta \ln d_{t+1}$$

with respect to the new constraints

$$\begin{aligned} c_t + s_t &= (1 - \tau_t)(1 - \sigma) w_t^s \left(1 - z n_t - \frac{\overline{x}}{n_{t-1}}\right) + \theta n_t, \\ d_{t+1} &= R_{t+1} s_t. \end{aligned}$$

Note that this tax policy will not change the wage equilibrium condition. Specifically, it still holds that $(1 - \sigma) w_t^s = w_t^u$. To close the model, the government budget balancing equation reads

$$\tau_t = \frac{\theta n_t}{(1 - \sigma) w_t^s \left(1 - z n_t - \frac{\overline{x}}{n_{t-1}}\right)}. \tag{35}$$

The household's optimal solutions are now given by

$$c_t = \frac{1}{1+\gamma+\beta}\left[(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)+\theta\underline{n}\right], \tag{36}$$

$$s_t = \frac{\beta}{1+\gamma+\beta}\left[(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)+\theta\underline{n}\right], \tag{37}$$

$$n_t-\underline{n} = \frac{\gamma}{1+\gamma+\beta}\frac{(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)+\theta\underline{n}}{(1-\tau_t)(1-\sigma)w_t^s z-\theta} \tag{38}$$

Proposition 2. *Given a positive child allowance $\theta > 0$ and $n_0 > \underline{n}$, we can state that $n_t > \underline{n}$ for all t.*

Proof. The fertility decision implies

$$\begin{aligned}
n_t-\underline{n} &= \frac{\gamma}{1+\gamma+\beta}\frac{(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)+\theta\underline{n}}{(1-\tau_t)(1-\sigma)w_t^s z-\theta}\\
&> \frac{\gamma}{1+\gamma+\beta}\frac{(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)}{(1-\tau_t)(1-\sigma)w_t^s z-\theta}\\
&> \frac{\gamma}{1+\gamma+\beta}\frac{(1-\tau_t)(1-\sigma)w_t^s\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)}{(1-\tau_t)(1-\sigma)w_t^s z}\\
&= \frac{\gamma}{(1+\gamma+\beta)z}\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right).
\end{aligned}$$

Assume that $n_0 > \underline{n}$. The limits when n_{t-1} is closed to $\underline{n}$ is

$$\lim_{n_{t-1}\to\underline{n}^+}(n_t-\underline{n}) > \lim_{n_{t-1}\to\underline{n}^+}\frac{\gamma}{(1+\gamma+\beta)z}\left(1-z\underline{n}-\frac{\overline{x}}{n_{t-1}}\right)=0$$

since $1-z\underline{n}-\overline{x}/\underline{n}=0$ by (7). In other words, $n_t > \underline{n}\ \forall t$ given $n_0 > \underline{n}$. □

This result implies that once the child allowance θ is implemented, starting from any initial fertility rates higher than $\underline{n}$, the fertility choice n_t is always greater than $\underline{n}$. The *nursing hell* problem can then be avoided. However, we need to verify the effects of this policy on other variables, especially welfare, since the lower net income may result in lower consumption and savings.

To do so, we assume that the rest of the model remains unchanged. Production is given by (1). The wage for skilled labor is still

$$w_t^s = (1-\alpha)Ak_t^\alpha$$

where $k_t \equiv K_t / L_t^s$. The laws of motion for the middle-aged population and capital per skilled labor are given by

$$N_t = n_{t-1} N_{t-1}, \tag{39}$$

$$k_t = \frac{s_{t-1}}{(1-\sigma)\phi_t n_{t-1}} \tag{40}$$

Given n_{t-1} and s_{t-1}, the rest of the model is determined as follows.

When $(1-\sigma)w_t^s = w_t^u = Ab$, some portion of unskilled labor will be allocated to final good production. This implies

$$w_t^s = (1-\alpha)Ak_t^\alpha = \frac{Ab}{(1-\sigma)}. \tag{41}$$

This implies

$$k_t = \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}.$$

Using (40) to eliminate k_t from this equation, we obtain the skilled worker ratio

$$\phi_t = \left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha} \frac{s_{t-1}}{(1-\sigma)n_{t-1}}.$$

On the other hand, when $(1-\sigma)w_t^s = w_t^u > Ab$, unskilled workers are not hired in the final good production and thus spend all their time on caring activities. This implies

$$\phi_t = 1 - zn_t - \frac{\overline{x}}{n_{t-1}}.$$

Using this information, the capital-skilled labor ratio becomes

$$k_t = \frac{s_{t-1}}{(1-\sigma)\left(1 - zn_t - \frac{\overline{x}}{n_{t-1}}\right) n_{t-1}} = \frac{s_{t-1}}{(1-\sigma)(n_{t-1} - zn_{t-1}n_t - \overline{x})}.$$

This implies a wage rate of

$$w_t^s = (1-\alpha)A\left[\frac{s_{t-1}}{(1-\sigma)(n_{t-1} - zn_{t-1}n_t - \overline{x})}\right]^\alpha, \tag{42}$$

which is a function of s_{t-1}, n_{t-1}, and n_t. In addition, the following condition must hold

$$k_t > \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}.$$

Unfortunately, the new model with policy intervention cannot be solved by hand. The dynamics of all endogenous variables and their steady states must be solved numerically. The procedure is stated as follows. Given $n_0 \in (\underline{n}, \frac{1+\sqrt{1-4\overline{x}z}}{2z})$, $s_0 > 0$, and a sufficiently small $\theta > 0$: [9]

1. Equations (42)(35)(38) uniquely determine n_t. Denote the solution as n_t^f. This is the fertility choice when unskilled workers spend their full time on caring activities.

2. Equations (41)(35)(38) uniquely determine n_t. Denote the solution as n_t^p. This is the fertility choice when unskilled workers split their time between caring for and working in the final goods sector.

3. The skilled worker ratio is determined by (34), specifically

$$\phi_t = \min\left\{\left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha}\frac{s_{t-1}}{(1-\sigma)n_{t-1}}, 1 - zn_t^f - \frac{\overline{x}}{n_{t-1}}\right\}.$$

   where if

$$\phi_t = \begin{cases} 1 - zn_t^f - \frac{\overline{x}}{n_{t-1}}, \text{ then } n_t = n_t^f, w_t^s = (1-\alpha)A\left[\frac{s_{t-1}}{(1-\sigma)(n_{t-1}-zn_{t-1}n_t^f-\overline{x})}\right]^{\alpha}, \\ \left[\frac{(1-\sigma)(1-\alpha)}{b}\right]^{1/\alpha}\frac{s_{t-1}}{(1-\sigma)n_{t-1}}, \text{ then } n_t = n_t^p, w_t^s = Ab/(1-\sigma). \end{cases} \tag{43}$$

The equilibrium of this economy is the sequence of $\{(n_t, \phi_t, w_t^s, \tau_t, s_t, k_t)\}_{t=1}^{\infty}$ where (n_t, ϕ_t, w_t^s) are uniquely solved using (43) (following the steps above) and (τ_t, k_t, s_t) are given by (35)(40)(37), respectively.

4.2 Numerical examples

We now present some numerical simulations to demonstrate the effectiveness of this policy. Recall that the nursing hell scenario may occur under the following parameters and initial conditions.

We can calculate $\underline{n} = 1.2753$. Without policy intervention, i.e., $\theta = 0$, fertility will converge to this value. This is shown previously in Table 2.

[9] This is because not every level of child allowance is feasible. A high allowance implies a higher tax rate, which can be destructive to the household's budget. For instance, with the parameters in Table 3, if we set $\theta = 0.037$, the required income tax would exceed the household's feasible income at some period, and thus it is impossible to implement. This is later verified in subsection 4.2.

Table 3: Parameter values for the *nursing hell* scenario

β	γ	α	σ	A	$\overline{x}$	b	z	s_0	n_0
0.99^{120}	0.3	0.36	0.4	6	0.95	0.2	0.2	0.1	3

On this path, dual caregiving slowly eats up all resources, leaving almost nothing for consumption and savings as fertility approaches $\underline{n}$. To steer the economy away from this outcome, we introduce a child allowance of $\theta = 0.01$. The government implements this policy at time $t = 1$, and maintains it at that level forever for every subsequent generation. While modest, this allowance is sufficient to prevent fertility from declining toward $\underline{n}$ while ensuring the tax burden does not exceed the household's feasible income. The dynamics are reported in Table 4.

Table 4: Dynamics when child allowance $\theta = 0.01$ is implemented.

t	τ_t	n_t	s_t	c_t	ϕ_t	k_t	u_t
0	0	3	0.10000				
1	0.04156	1.7061	0.09460	0.3160	0.3401	0.1633	-1.7868
2	0.05030	1.4633	0.06703	0.2239	0.1505	0.6166	-2.5769
3	0.07815	1.3730	0.04047	0.1352	0.0762	1.0074	-3.4598
4	0.13072	1.3122	0.02348	0.0784	0.0417	1.1874	-4.3356
5	0.21876	1.3034	0.01382	0.0462	0.0243	1.2206	-5.1429
6	0.35168	1.3024	0.00854	0.0285	0.0153	1.1647	-5.8279
7	0.52564	1.3132	0.00571	0.0191	0.0107	1.0477	-6.3407
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	0.54756	1.3104	0.00551	0.0184	0.0129	0.5413	-6.2075

As we can see, the economy is saved, but maintaining the child allowance ultimately requires a high labor income tax. In our case, an allowance of 0.01 per child necessitates a steady-state income tax of 54% to sustain the fertility of 1.3104 (about 3% higher than the *nursing hell* case). More importantly, this policy brings forth Pareto improvement for all generations. To see this, compare generation 1 without policy (in Table 2) and with policy (in Table 4). Although higher taxes reduce individual consumption and savings, the rise in fertility can offset this negative effect, resulting in a net increase in welfare. The improvement in fertility steers the economy away from the *nursing hell* outcome and places it on a sustainable path. As a result, the skilled worker ratio and consumption never

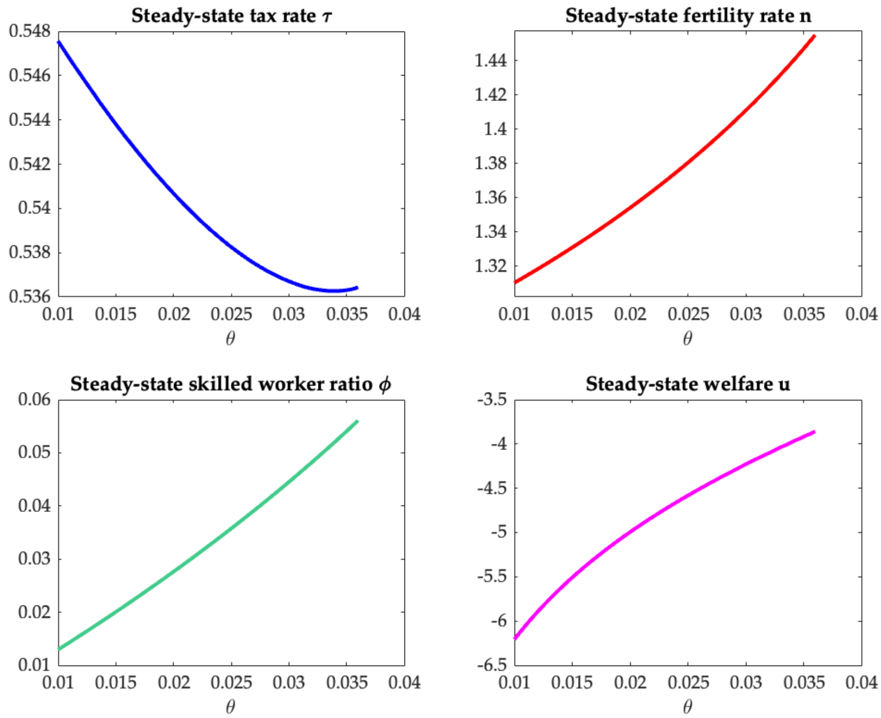


Figure 3: Steady-state outcomes with different values of child allowances.

go asymptotically zero, which makes their welfare significantly better than the case without child allowances.

We now examine the steady-state outcomes under different levels of child allowances. This exercise serves two purposes. First, the previous example shows that child allowance can improve intergenerational welfare. In that case, we want to know whether there is an optimal child allowance that brings forth the best steady-state welfare for this economy. Second, we have noted that not every child allowance policy is feasible. The government cannot implement a too generous child allowance policy as doing so may require an unreasonably high tax rate, i.e., τ can exceed 1. In what follows, we experiment with the policy range of $\theta \in [0.01, 0.04]$ and show the corresponding steady-state outcomes in Fig.3.

Fig. 3 shows that a higher child allowance leads to higher fertility in the steady state. Thanks to the increased sibling size, the reduced dual caregiving burden allows agents to supply more skilled labor, thereby increasing the net income per middle-aged agent. This explains why the steady-state tax rate can decline as child allowances increase. According to eq.(35), this outcome occurs when the additional income generated outweighs the fiscal cost of higher fertility (the gains in the denominator are greater than the gains in the numerator). Furthermore, since agents derive utility from consumption and altruism, these rises in net income and fertility result in higher steady-state welfare. It is also worth noting that, although we simulate θ values up to 0.04, it turns out that the maximum feasible θ that can be implemented is 0.036. As a result, this is the optimal policy. Any child allowance above this threshold cannot be sustained.

5 What happens if $\overline{x}z > 1/4$?

So far, the possibility that $\overline{x}z > 1/4$ has been excluded by assumption. We are now in a position to reveal the reason for this. In a nutshell, such an economy will inevitably fail.

Proposition 3. *When $\overline{x}z > 1/4$, the manpower needed to care for children and the elderly will exceed the labor supply of the middle-aged generation in finite periods.*

Proof. If the middle-aged generation in period t invests all of their endowed labor in child and elder care, then the following equation holds:

$$1 - zn_t - \overline{x}/n_{t-1} = 0$$

or

$$n_t = (1 - \overline{x}/n_{t-1})/z. \tag{44}$$

When $\overline{x}z > 1/4$, on a plane with n_{t-1} on the horizontal axis and n_t on the vertical axis, the graph for (44) lies below the 45-degree line, i.e.

$$n_t = (1 - \overline{x}/n_{t-1})/z < n_{t-1},$$

meaning that starting from any initial value, the fertility rate takes on a negative value after a finite time (see Figure 4). In other words, some periods after the beginning of the economy, all of the labor imparted to the middle-aged generation combined will be insufficient to care for their parent's generation. □

This proposition asserts that even if all labor is invested in caregiving, there will be a generation in a few periods that will not be able to care for their own parents' generation. If some portion of the labor force were

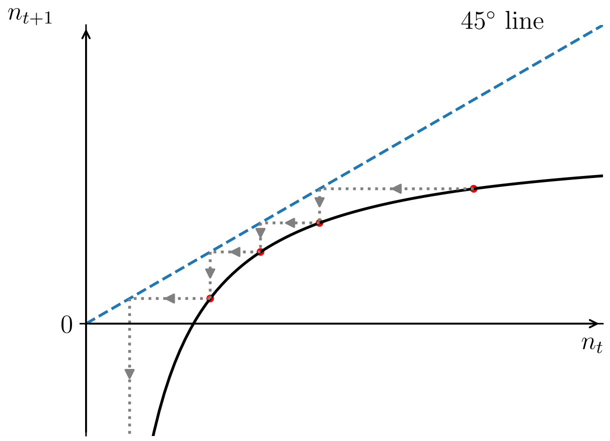


Figure 4: Dynamics of n_t when $\overline{x}z > 1/4$.

invested in productive activities, such a generation would emerge much earlier.

More critically, this failure cannot be avoided by the policy examined in the previous section. That policy was to raise the birth rate of the middle-aged generation, thereby preventing the economy from descending into a nursing hell. However, the above proposition implies that the maximum feasible birth rate is already chosen in each period, leaving no room for the policy to be effective. In the first place, that policy was effective because the following equation holds for n slightly larger than $\underline{n}$:

$$1 - zn - \overline{x}/n > 0,$$

which is possible only when $\overline{x}z \leq 1/4$. Thus, the burden of care beyond a certain critical level makes the very survival of the economy difficult.

6 Conclusion

This paper has examined the impact of the dual care burden on economic growth in the framework of an overlapping generations model and concluded that a heavy burden negatively impacts economic growth. More importantly, when the care burden exceeds a certain critical level, even the survival of the economy is at stake. The dual care burden must be maintained at a sufficiently low level to achieve sustainable growth.

These conclusions raise the following natural question: why has human society survived to the present day, although the survival condition identified in this paper seems too restrictive to be met? One plausible answer would be that humans have had a short life span for a long time, and not many people could reach old age. This would have contributed greatly to keeping the burden of elder care at a low level. Today, however, these circumstances have drastically changed, as Wilmoth (2011) wrote:

> Life expectancy has been increasing not only in industrialized societies but also around the world. According to estimates by the United Nations, life expectancy at birth for the world as a whole has risen from around 46 years in 1950 to approximately 68 years in 2009. During this same time interval, life expectancy at birth has increased from 65 to 77 years for the more developed regions and from 40 to 66 years for the less developed regions. Even the least developed countries have experienced a rise in life expectancy at birth over this period, from 35 to 57 years. (pp.156-158)

This worldwide increase in longevity is the fruit of scientific progress since the Industrial Revolution. Science has fulfilled our wish to live longer. In exchange for that, it has created a new burden of dual caregiving.

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A Proof of proposition 1

Proof. Equation (27) implies that its dynamical system has two stationary points, $\underline{n}$ and $\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}$. The relationship between them depends on the value of $\overline{x}z$. Specifically, when $\overline{x}z \in (0, \eta)$,

$$\underline{n} < \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}. \tag{45}$$

When $\overline{x}z = \eta$,

$$\underline{n} = \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}. \tag{46}$$

When $\overline{x}z \in (\eta, 1/4]$,

$$\underline{n} > \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}. \tag{47}$$

To see this fact, note that comparing $\underline{n}$ and $\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}$ is the same as comparing $\frac{(z\underline{n})^2}{\overline{x}z}$ and $\frac{\gamma}{1+\gamma+\beta}$. For example, (45) is equivalent to the following inequality:

$$\frac{(z\underline{n})^2}{\overline{x}z} < \frac{\gamma}{1+\gamma+\beta}. \tag{48}$$

Using (7), we can rewrite the LHS of (48) as

$$\frac{(z\underline{n})^2}{\overline{x}z} = \frac{(1-\sqrt{1-4\overline{x}z})^2}{4\overline{x}z}.$$

By differentiating it with $\overline{x}z$, we have

$$\frac{d}{d(\overline{x}z)}\frac{(z\underline{n})^2}{\overline{x}z} = \frac{1-\sqrt{1-4\overline{x}z}}{4(\overline{x}z)^2}\left(\frac{1}{\sqrt{1-4\overline{x}z}} + \sqrt{1-4\overline{x}z} - 1\right).$$

This derivative is positive since the arithmetic-geometric mean relationship implies that

$$\frac{1}{\sqrt{1-4\overline{x}z}} + \sqrt{1-4\overline{x}z} - 1 \geq 2\left(\frac{1}{\sqrt{1-4\overline{x}z}} \cdot \sqrt{1-4\overline{x}z}\right)^{1/2} - 1 = 1 > 0.$$

Thus, we can state that $(z\underline{n})^2/\overline{x}z$ is an increasing function of $\overline{x}z$. In addition,[10]

$$\lim_{\overline{x}z \to 0} \frac{(z\underline{n})^2}{\overline{x}z} = 0, \qquad \text{and} \qquad \lim_{\overline{x}z \to 1/4} \frac{(z\underline{n})^2}{\overline{x}z} = 1.$$

These results jointly mean that there is a unique value of $\overline{x}z$ such that

$$\frac{(z\underline{n})^2}{\overline{x}z} = \frac{(1-\sqrt{1-4\overline{x}z})^2}{4\overline{x}z} = \frac{\gamma}{1+\gamma+\beta}.$$

By solving this equation for $\overline{x}z$, we can obtain the value of η. Since $(z\underline{n})^2/\overline{x}z$ is incresing with $\overline{x}z$, when $\overline{x}z < \eta$, the following inequality is true:

$$\frac{(z\underline{n})^2}{\overline{x}z} < \frac{\gamma}{1+\gamma+\beta},$$

which is equivalent to (45). Likewise, when $\overline{x}z > \eta$, the following is true:

$$\frac{(z\underline{n})^2}{\overline{x}z} > \frac{\gamma}{1+\gamma+\beta},$$

which is equivalent to (47). All that is needed to obtain the desired result is to check the positional relationship between the graph of (27) and the 45-degree line in the coordinate plane with n_t on the horizontal axis and n_{t+1} on the vertical axis. To accomplish this task, let us rewrite (27) as follows:

$$n_t - n_{t-1} = -\frac{n_{t-1} - \underline{n}}{n_{t-1}} \left[n_{t-1} - \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}} \right]. \tag{49}$$

When the RHS of (49) is positive, the graph of (27) is above the 45-degree line; when it is negative, it is below. When $\overline{x}z \in (0,\eta)$, the RHS of (49) is positive for $n_{t-1} \in \left(\underline{n}, \frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}\right)$ and negative for $n_{t-1} \in \left(\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}, +\infty\right)$, which means that n_t is approaching $\frac{\gamma\overline{x}}{(1+\gamma+\beta)z\underline{n}}$, independent of its initial value, as shown in Figure 2a. On the other hand, when $\overline{x}z \in [\eta, 1/4]$, the RHS of (49) is negative for $n_{t-1} \in (\underline{n}, +\infty)$, which means that starting from any point larger than $\underline{n}$, n_t is decreasing to $\underline{n}$ over time, as shown in Figure 2b. □

[10] To derive the first equation, use L'Hôpital's theorem.

B Note on division of labor

When the economy follows the *sustainable growth* path, characterized by $\overline{x}z < 1/4$, we can determine the steady-state division of labor – specifically, whether unskilled workers allocate some of their labor to final goods production or dedicate all of it to caregiving. In the former case, they function as part-time caregivers, while in the latter, they become full-time caregivers. Regardless of either outcome, the steady-state fertility n^* is

$$n^* = \frac{\gamma\overline{x}}{(1+\beta+\gamma)z\underline{n}}.$$

Assume that the steady state ϕ^* is to have unskilled workers specialized in the dual caregiving duty, it is characterized by the following steady-state equations

$$\begin{cases} n^* &= \dfrac{\gamma\overline{x}}{(1+\beta+\gamma)z\underline{n}}, \\ s^* &= \dfrac{(1-\sigma)(1-\alpha)\beta A}{1+\beta+\gamma}(k^*)^\alpha(1-z\underline{n}-\overline{x}/n^*), \\ \phi^* &= 1-zn^*-\overline{x}/n^*, \\ k^* &= \dfrac{s^*}{(1-\sigma)\phi^* n^*}. \end{cases}$$

Combining all these equations yields

$$k^* = \left[\frac{\beta(1-\alpha)A}{1+\beta+\gamma}\cdot\chi(n^*)\right]^{\frac{1}{1-\alpha}}. \tag{50}$$

where $\chi(n^*)$ is given by:

$$\chi(n^*) = \frac{1-z\underline{n}-\overline{x}/n^*}{(1-zn^*-\overline{x}/n^*)n^*},$$

Since the inequality (31) must hold in this equilibrium, equations (31) and (50) jointly imply that

$$\left[\frac{\beta(1-\alpha)A}{1+\beta+\gamma}\cdot\chi(n^*)\right]^{\frac{1}{1-\alpha}} > \left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{1/\alpha}$$

or

$$A > \frac{1+\beta+\gamma}{(1-\alpha)\beta\chi(n^*)}\left[\frac{b}{(1-\sigma)(1-\alpha)}\right]^{\frac{1-\alpha}{\alpha}}.$$

This result shows that when labor productivity is sufficiently large, unskilled workers can spend all their time on caregiving duties. Otherwise, when labor productivity is sufficiently low, unskilled workers work partially on the market while simultaneously fulfilling the dual care responsibility at home.