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An Analytical Model of Search and Bargaining with Divisible Money*

Kazuya Kamiya[†] and So Kubota[‡]

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Abstract

We propose a standard search and bargaining model with divisible money, in which only the random matching market opens and the generalized Nash bargaining settles each trade. Assuming fixed production costs, we analytically characterize a tractable equilibrium, called a *pay-all equilibrium*, and prove its existence. Each buyer pays all the money holding as a corner solution to the bargaining problem and each seller produces a positive amount of goods as an interior solution. The bargaining power parameter affects the distribution of the money holdings and possibly induces economic inefficiency. We propose a redistributional monetary transfer that adjusts the bargaining outcome and improves the allocation efficiency. Moreover, we analyze a temporary expansion of the money supply that increases social welfare through a redistribution.

Keywords: Money, Search, Bargaining, Distributions

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1 Introduction

We propose a search and bargaining model with money and study a tractable equilibrium with simple distributions of money holdings. The model is a straightforward extension of basic search and bargaining models with indivisible money, such as Trejos and Wright (1995) and Shi (1995) to divisible money. In this model, both money and goods are divisible and traded in an environment of only random matching and Nash bargaining. This formulation has been well-known in the literature since it was first conjectured by Trejos and Wright (1995). However, to the best of our knowledge, no study has yet succeeded in analytically characterizing the monetary equilibria because of the technical difficulties in dealing with the distribution of money holdings. We find that, by assuming fixed production costs, the model generates a simple but non-degenerate distribution of money holdings. We analytically characterize the equilibria, prove their existence, and derive the efficiency conditions of the bargaining power parameter. Our model also has implications for redistributive fiscal and monetary policies.

Originating from Kiyotaki and Wright (1989), search theory provides a solid microfoundation of money and has implications for real-world economic phenomena. However, its straightforward extension to divisible money makes its distribution non-degenerate, which is hard to monitor as a state variable. To overcome this problem, workhorse (or the so-called third-generation) models impose the large household assumption (Shi (1997)) and centralized markets (Lagos and Wright (2005)) to make the distribution degenerate. While these extended models are attractive in terms of tractability and due to their rich applications, it is still worth considering basic search and bargaining models. These elementary models shed light on theoretical properties and policy applications in the search market that may be hidden by tractability assumptions.

We thus consider a standard search model in which agents randomly match and exchange goods with money through Nash bargaining. There is no assumption to ensure the degenerate distribution of money holdings. Instead, by introducing fixed production costs, we construct a particular type of tractable equilibrium that satisfies the *pay-all* property.

In each Nash bargaining problem, the amount of monetary payment is solved as a corner solution, that is, the buyer pays all the money holding. By contrast, the amount of goods sold is determined as an interior solution. Then, a simple stationary non-degenerate distribution of money holdings arises: there is a mass of agents with zero money holding, while the others save sufficiently large amounts of money to overcome the fixed costs to purchase goods. Agents alternately move between the two states, and this sorting helps keep track of the distribution of money holdings. Besides Nash bargaining, this equilibrium also holds under Kalai (1977b)'s proportional solution.

The equilibrium has an inefficiency caused by the bargaining power parameter between the buyer and seller, in the manner of the Hosios (1990) condition. Our model suggests a holdup problem that distorts the buyer's incentive to save money as in Lagos and Wright (2005). In addition to these intra-bargaining effects, in our model, bargaining power affects the distribution of money holdings. This new channel makes a non-monotonic relationship between social welfare and bargaining power. We thus propose redistributive monetary transfers that mitigates this bargaining power inefficiency while maintaining the budget balance. Moreover, we study monetary expansions and propose an analytical result of an effective short-term monetary policy associated with the redistribution of money holdings. This proportional neutrality and distributional effects are in line with Wallace (2014)'s conjecture.

Some models in the literature have succeeded in characterizing the non-degenerate distribution of money holdings. The first contribution is that of Green and Zhou (1998), who consider a random-matching model with divisible money in which each exchange is closed by a take-it-or-leave-it offer; this is a special case of Nash bargaining. The key result is the real indeterminacy of stationary equilibria: a continuum of steady states exists and each holds a different real allocation.[1] Although the result is theoretically appealing, this property hinders applied studies. Zhu (2005) also considers a model with a take-it-or-leave-it offer and constructs the equilibrium of divisible money as the limit of the indivisible money equilibrium. Another important model is that of Menzio et al. (2013), which eliminates one-to-one Nash bargaining and instead assumes a competitive search environment. They construct a block recursive equilibrium that creates a simple transitional process of agents on the non-degenerate distribution. In a more recent study, Rocheteau et al. (2021) analyze

[1] The general framework is constructed by Kamiya and Shimizu (2006). The indeterminacy emerges even if goods are divisible. See Kamiya and Shimizu (2007, 2011, 2013), and Kubota (2019) for the conditions of this indeterminacy.

a discrete distribution caused by a delayed money-holding adjustment based on the Lagos-Wright model. The use of numerical methods of Molico (2006), Chiu and Molico (2010), and Chiu and Molico (2021) is also noteworthy. Finally, Camera and Corbae (1999) consider a model in which agents can hold countable amounts of money. The novelty of our approach, by contrast, arises from (i) the tractable equilibria constructed only on a random-matching market with Nash bargaining, (ii) the analytical characterization and existence of equilibria, and (iii) the policy implications of bargaining power inefficiency.

The remainder of this paper proceeds as follows. Section 2 introduces the economic environment. Section 3 illustrates the pay-all equilibrium and its properties, and demonstrates how a perturbation selects a unique equilibrium. Section 4 proves the existence of the equilibria. Section 5 investigates the economic inefficiency induced by bargaining power on social welfare, and Section 6 considers a redistributive policy to improve allocation. Section 7 discusses monetary expansion, and Section 8 considers two related problems: the choice of the bargaining solutions and consistency with axiomatic Nash bargaining. Finally, Section 9 concludes the paper.

2 Economic environment

Time is discrete and the time horizon is infinite, as denoted by $t = 1, 2, \cdots$. There is a continuum of agents with measure one. Each agent discounts her future payoff with a discount factor $\beta \in (0, 1)$, and each can produce divisible and non-storable goods. This incurs disutility following a cost function:

$$u^s(x) = \begin{cases} -d - cx & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where x is the amount of production, $d > 0$ is the fixed cost, and $c \in (0, 1)$ is the unit cost. To eliminate the double-coincidence of wants, we assume that each agent cannot consume her production goods. However, they can consume some others' production goods with a temporal utility function:

$$u^b(x) = \begin{cases} k + x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $k > 0$ is a constant. Theoretically, k is introduced for the individual rationality constraint so that the discounted sum of utility is always positive in equilibrium. Specifically, the discounted value of agents having no money will be shown as $(ck - d)/\beta(1 - c)$ in Proposition 1. Hence, $k > 0$ is necessary for an agent without money to participate in the market. We later discuss how these assumptions about utility and cost functions ensure the tractability of equilibrium.

This good might be considered partially indivisible, that is, production starts with an indivisible unit and then uses a fully divisible one. First, the fixed costs work as in Green and Zhou (1998), where the indivisibility makes a kind of sorting of money holdings and turns the distribution discrete. Similarly, in our model, the fixed costs also make a small amount of money less valuable and divide the distribution into two parts. The variable costs are related to the decentralized market in Lagos and Wright (2005). In our model, variable costs are small enough to derive pay-all property. Interestingly, the Lagos-Wright model also shows the equilibrium that buyers spend all money holdings.

There is another good, money, which is perfectly divisible and storable. The total supply in the economy is fixed at $M > 0$. Each agent's money holding, m, is a nonnegative real number with an upper-bound $\overline{m}$, that is, $m \in [0, \overline{m}]$. This assumption allows us to ignore a behavior of very rich agents who exist only off-path and simplifies the proof of equilibrium existence.[2]

The timeline for each period is as follows. At the beginning of each period, agents observe the current economy-wide distribution of money holdings. Then, pairwise random matching occurs with probability $2\alpha \in (0, 1)$. In each matching, each agent observes their partner's money holding. All matchings are held with a single-coincidence of wants: one agent becomes a seller with probability $\frac{1}{2}$ and the other becomes a buyer. Subsequently, the seller and buyer negotiate the monetary payment and the amount of goods exchanged

[2]Zhu (2005) also assumes the upper-bound for the proof of the existence of monetary equilibrium and Zhou (1999) derives the endogenous emergence of the upper-bound.

following generalized Nash bargaining with the buyer's bargaining power, $\theta \in (0,1)$.[3] At the end of each period, the matching resolves and proceeds to the next period. Let H be the distribution of money holdings (a Borel measure) on $\mathbb{R}_+$. In a meeting between a buyer and a seller with money holdings m_b and m_s, respectively, Nash bargaining decides the amounts of goods $x(m_b, m_s, H)$ and of monetary payment $p(m_b, m_s, H)$ in the trade. No trade, $x(m_b, m_s, H) = p(m_b, m_s, H) = 0$, is also a solution when bargaining surpluses cannot be nonnegative.

For the definition of the stationary distribution, we introduce the following two sets. First, for a Borel set $D \subset \mathbb{R}_+$, the set of agents who meet an agent (a buyer) with m_b and move to D after the trade is denoted by $Q(m_b, D) = \{m_s \mid m_s + p(m_b, m_s, H) \in D\}$. This includes trade cases where $p(m_b, m_s, H) > 0$, and no-trade cases where $p(m_b, m_s, H) = 0$ and $m_s \in D$. Second, the set of agents who meet an agent (a seller) with m_s and move to D after the trade is denoted by $R(m_s, D) = \{m_b \mid m_b - p(m_b, m_s, H) \in D\}$, including those who meet a partner but do not trade and remain in D.

Definition 1. *Let v be a function on $\mathbb{R}_+$. A pair (H, v) is called a stationary monetary equilibrium if*

1. *The Bellman equation is consistently constructed as*

$$v(m) = \alpha \int_0^{\overline{m}} \big[u^b\big(x(m, m_s, H)\big) + \beta v\big(m - p(m, m_s, H)\big)\big]\, dH(m_s) + \alpha \int_0^{\overline{m}} \big[u^s\big(x(m_b, m, H)\big) + \beta v\big(m + p(m_b, m, H)\big)\big]\, dH(m_b) + (1 - 2\alpha)\beta v(m),$$

2. *H is a stationary distribution of the process under $p(m_b, m_s, H)$. That is, for all Borel sets $D \subset \mathbb{R}_+$, (i) $H(Q(m_b, D))$ and $H(R(m_s, D))$ are measurable functions[4] of m_b and m_s, and (ii) the outflow from D, expressed as $2\alpha H(D)$, is equal to the inflow to D expressed as*

$$2\alpha H(D) = \alpha \left(\int_0^{\overline{m}} H(Q(m_b, D))\, dH(m_b) + \int_0^{\overline{m}} H(R(m_s, D))\, dH(m_s) \right).$$

[3]Section 8 discusses the consistency of the maximization of the Nash product in our model with the axiomatic Nash bargaining solution; specifically, we discuss the problem related to the convexity of the bargaining set and show its consistency.

[4]Note that the condition of measurability is satisfied in the pay-all equilibrium defined in the following section.

3. $x(m_b, m_s, H) \geq 0$ *and* $p(m_b, m_s, H) \geq 0$ *solve each Nash bargaining problem that*

   - *if there exists* (x, p) *such that* $x > 0$, $0 \leq p \leq m_b$, *and both the buyer and seller's surpluses are nonnegative, that is,* $k + x + \beta\big(v(m_b - p) - v(m_b)\big) \geq 0$ *and* $-d - cx + \beta\big(v(m_s + p) - v(m_s)\big) \geq 0$, *then the trade* $(x(m_b, m_s, H), p(m_b, m_s, H))$ *is determined by*

$$(x(m_b, m_s, H), p(m_b, m_s, H)) = \\ \arg\max_{x,p} \left[k + x + \beta\big(v(m_b - p) - v(m_b)\big)\right]^{\theta} \left[-d - cx + \beta\big(v(m_s + p) - v(m_s)\big)\right]^{1-\theta} \\ \textit{s.t.} \quad x > 0, \quad 0 \leq p \leq m_b$$

   - *Otherwise,* $x(m_b, m_s, H) = p(m_b, m_s, H) = 0$.

4. $v(m) > 0$ *for all* $m \geq 0$ *and* $v(m)$ *is strictly increasing.*

The last condition represents the individual rationality and positive equilibrium value of fiat money.

3 Pay-all equilibrium

3.1 The definition and the properties of pay-all equilibrium

Here, we focus on a particular subset of the stationary monetary equilibrium: the *pay-all* equilibrium. Specifically, we show that this equilibrium is sufficiently tractable to prove existence and investigate its characteristics.

Definition 2. *A stationary monetary equilibrium is called a pay-all equilibrium if the following two conditions are satisfied. At each Nash bargaining,*

1. *if the bargaining is agreed with* $x(m_b, m_s, H) > 0$, *the monetary payment is binding at* $p(m_b, m_s, H) = m_b$, *and*

2. *if both agents hold strictly positive amount of money in the support of* H, *the bargaining is disagreed, that is,* $x(m_b, m_s, H) = p(m_b, m_s, H) = 0$.

We call this equilibrium pay-all because, as in the first condition, each buyer spends all the money holding. The second condition means that a seller holding $m > 0$ in the support of the equilibrium distribution of money holdings never sells goods. The two conditions in Definition 2 are collectively called the *pay-all property.*

Let the support of distribution H be $\{0\} \cup [\underline{z}, \overline{Z}]$ and $H(0) \in (0,1)$ the measure of agents holding no money, where $0 < \underline{z} \leq \overline{Z}$ and H satisfies

$$\int_{\underline{z}}^{\overline{Z}} dH(m) = 1 - H(0), \tag{1}$$

$$\int_{\underline{z}}^{\overline{Z}} m dH(m) = M. \tag{2}$$

Then, $M/(1 - H(0))$ is the average money holding of the agents with $m > 0$. An example of H is shown in Figure 1, where the density function for $m \in [\underline{z}, \overline{Z}]$ is bell-shaped. However, the pay-all equilibrium holds with any distribution on a finite interval, as shown in the next section. For example, a uniform distribution is included. In an extreme case, a degenerate distribution with $\underline{z} = \overline{Z}$ is also possible. In the pay-all equilibrium, only agents holding $m = 0$ ($m \in [\underline{z}, \overline{Z}]$) sell (buy) goods to the sellers (buyers). Then, there are two types of transitions in the distribution of money holdings: (i) buyers spend all the money holding and move to $m = 0$ and (ii) sellers receive money and move to $m \in [\underline{z}, \overline{Z}]$. Given the second condition of Definition 2, there is no trade in a meeting where both agents hold $m \in [\underline{z}, \overline{Z}]$. We will show that, in this case, the seller's surplus is negative due to fixed costs d.

In this section, we also focus on on-path trades. Specifically, we consider the case in which $m_b \in [\underline{z}, \overline{Z}]$ and $m_s = 0$. Therefore, we denote $x(m) = x(m, 0, H)$ and $p(m) = p(m, 0, H)$, where the buyer holds $m_b = m$.

Given corner solution $p(m) = m$, the Nash bargaining problem derives the amount of goods by

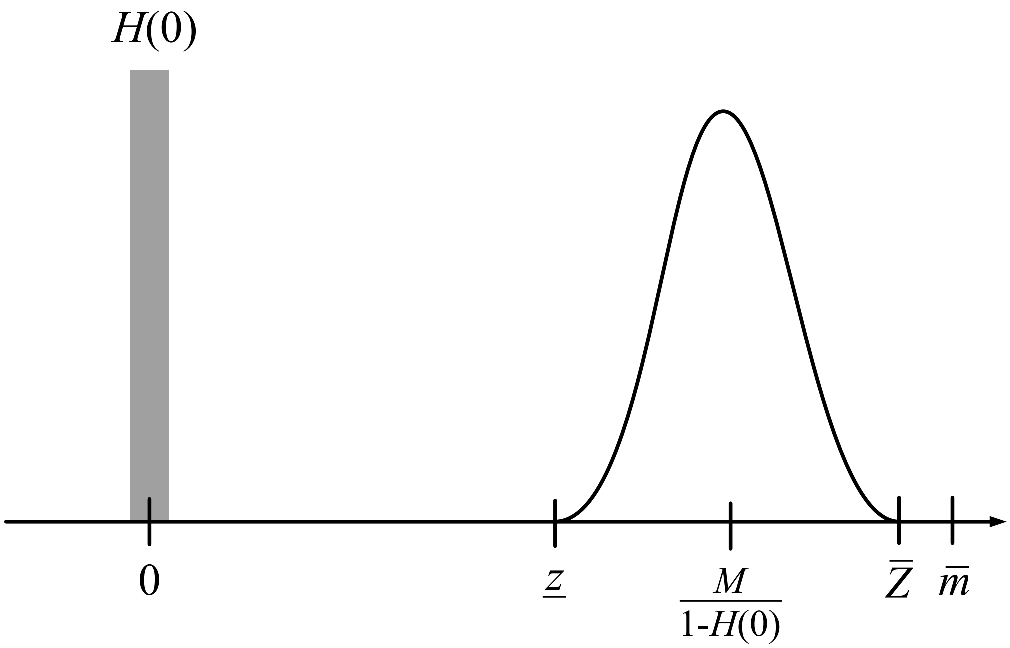
$$x(m) = \arg\max_x \left[k + x + \beta\big(v(0) - v(m)\big)\right]^{\theta} \left[-d - cx + \beta\big(v(m) - v(0)\big)\right]^{1-\theta}, \tag{3}$$

and then, its first-order condition is

$$(1 - \theta)c \left[k + x + \beta\big(v(0) - v(m)\big)\right] = \theta \left[-d - cx + \beta\big(v(m) - v(0)\big)\right]. \tag{4}$$

The total surplus is divided according to relative bargaining power θ, marginal costs c, and marginal utility 1. Note that the second-order condition is satisfied globally since, by taking

Figure 1: An example of the distribution of money holdings in the pay-all equilibrium



the logarithm of the Nash product in (3), the objective function is converted into a concave function.[5] Equation (4) can be rewritten as

$$cx(m) = \beta[\theta + (1-\theta)c]\big(v(m) - v(0)\big) - [(1-\theta)ck + \theta d]. \tag{5}$$

The on-path Bellman equation, that is, the case of $m \in \{0\} \cup [\underline{z}, \overline{Z}]$ in the pay-all equilibrium, is written as

$$v(m) = \alpha H(0)\big(k + x(m) + \beta v(0)\big) + \big(1 - \alpha H(0)\big)\beta v(m) \qquad \text{for } m \in [\underline{z}, \overline{Z}], \tag{6}$$

$$v(0) = \alpha \int_{\underline{z}}^{\overline{Z}} \big(-d - cx(m) + \beta v(m)\big) dH(m) + \big[1 - \alpha\big(1 - H(0)\big)\big]\, \beta v(0). \tag{7}$$

The full Bellman equation, including off-path money holdings, $m \notin \{0\} \cup [\underline{z}, \overline{Z}]$, is given in the next section. Equation (6) is the Bellman equation for an agent holding a positive amount of money. This agent meets another agent with probability 2α. Subsequently, the

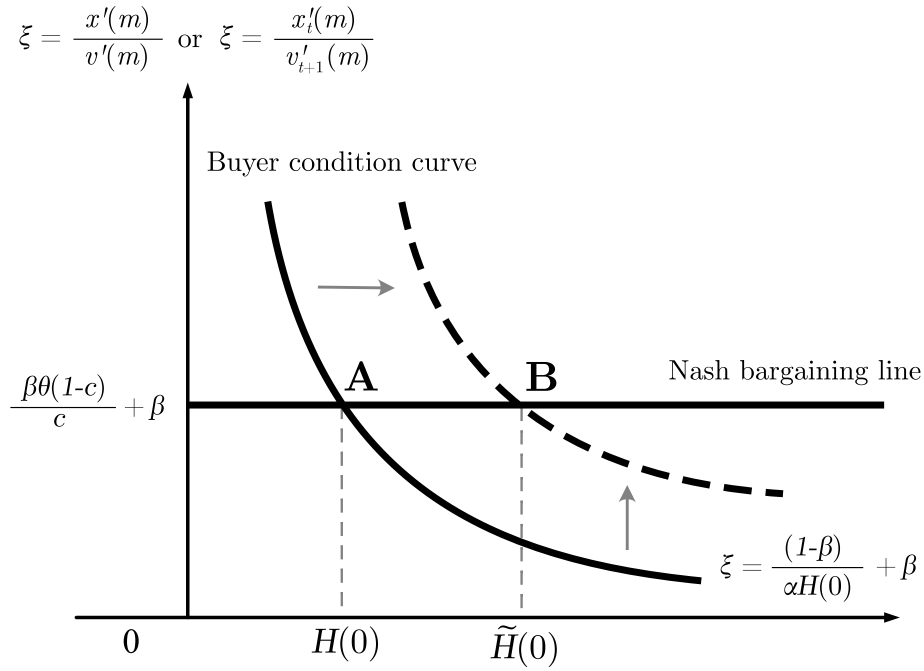
[5] If either the buyer's or seller's surplus is zero, the function value is undefined. This does not occur under the conditions necessary for the existence of a pay-all equilibrium (see the proof of Lemma 7).

first agent becomes a buyer with probability $1/2$ and the counterpart is no money holder with probability $H(0)$. After bargaining, this agent obtains surplus $k + x(m) + \beta v(0)$, pays all the money holdings, and moves to $m = 0$ on the distribution of money holdings H in the next period. The Bellman equation for an agent with $m = 0$ is (7). This agent meets another agent and becomes a seller with probability α. The probability that the counterpart has $m \in [\underline{z}, \overline{Z}]$ is $1 - H(0) = \int_{\underline{z}}^{\overline{Z}} dH(m)$. Then, this agent obtains surplus $-d - cx(m) + \beta v(m)$, where the amount of production $x(m)$ and the next period value $v(m)$ depend on the money holding of counterpart $m \in [\underline{z}, \overline{Z}]$. After the trade, this agent moves from 0 to m in the support of H.

Discussion about assumptions. The existence of a pay-all equilibrium depends on the shapes of the utility and cost functions. Fixed costs d separate the equilibrium distribution between $m = 0$ and a sufficiently large positive amount $m \in [\underline{z}, \overline{Z}]$. Consider a seller who meets a buyer holding only a small amount of money. The seller will decline production because the profit will not cover the fixed costs. Then, in a stationary equilibrium, no agents hold such a small amount of money. This property lets agents play only one role, either buyer or seller, depending on money holding m. Therefore, the transition of the distribution of money holdings becomes simple: in each period, the same number of agents is exchanged between $\{0\}$ and $[\underline{z}, \overline{Z}]$. The linearity of the utility function and variable costs are also crucial for tractability. These assumptions induce buyers' incentives for pay all. By linearity, the marginal utility in the current period is always higher than that in the next period because of the discount factor and search friction. Therefore, buyers have no incentives to save money.

Our pay-all equilibrium is similar to Rocheteau et al. (2018)'s full-depletion equilibrium. In this study, if the cost of having a real balance is higher than the cost of insuring the future consumption, then the agent uses all real balances. This condition does not necessarily hold because of the nonlinearity of the utility function. However, if the inflation rate is sufficiently high, the condition is satisfied and full depreciation occurs. In our case, due to the linearities of the utility and cost functions, the pay-all (full-depletion) property always holds. Rocheteau et al. (2018) also use the linear utility function, where the coefficient changes stochastically, to simply express the probability of full-depletion and to analyze the hot potato effect. In our case, the coefficient is not stochastic and pay-all always occurs.

Figure 2: Determination of $H(0)$



The measure of zero-money holders. Here, we derive the measure of potential sellers $H(0)$. In what follows, we assume the differentiability of $v(m)$ and $x(m)$ for $m \in [\underline{z}, \overline{Z}]$. In Section 4, we prove the existence of pay-all equilibria with differentiable functions. To derive $H(0)$, we define $\xi(m) \equiv x'(m)/v'(m)$ for $m \in [\underline{z}, \overline{Z}]$.

Lemma 1. *In a pay-all equilibrium, $\xi(m)$ is constant, denoted by ξ, as follows.*

$$\xi(m) = \xi \equiv \frac{\beta\theta(1-c)}{c} + \beta, \tag{8}$$

and

$$H(0) = \frac{(1-\beta)c}{\alpha\beta\theta(1-c)}. \tag{9}$$

Proof. Equation (8) is derived from the first-order derivative of (4). Similarly, by the first-order derivative of (6),

$$\xi(m) = \frac{1-\beta}{\alpha H(0)} + \beta. \tag{10}$$

Equilibrium $H(0)$ is determined such that $\xi(m)$ in (8) and (10) are equal. □

$H(0)$ is the measure of agents holding no money to maintain stationary equilibrium. It is determined to equate ξ derived by two independent equations (8) and (10). This is represented by intersection A in Figure 2. ξ is the ratio between the current and future value of money. The numerator of ξ, $x'(m)$, is interpreted as the buyer's returns (in the utility term) from holding an additional unit of money in the current bargaining. The denominator, $v'(m)$, represents the future marginal value of money.

First, the horizontal line in Figure 2 is derived from the Nash bargaining's first-order condition (4). ξ is independent of $H(0)$ because the Nash bargaining decides intratemporal allocation given the successful matchings. Next, the decreasing curve is derived by the buyer's Bellman equation (6). By search friction, a buyer's trade chance arrives infrequently. As $H(0)$ increases, the buyer's meeting probability rises, which results in an increase in the marginal value of money $v'(m)$. $H(0)$ is determined by equating ξ in these two conditions.

To clarify the role of $H(0)$, assume that the population of no money holders is $\tilde{H}(0) \neq H(0)$. Under the pay-all property, $\tilde{H}(0)$ remains unchanged over time. Therefore, the economy cannot move $\tilde{H}(0)$ back to $H(0)$. Instead, curves must shift to equate ξ in Figure 2. Below, we discuss that this adjustment is achieved by making the economy non-stationary. We denote $v(m)$ and $x(m)$ in period t as $v_t(m)$ and $x_t(m)$, respectively. Figure 2 now includes the non-stationary conditions and shows the case of $\tilde{H}(0) > H(0)$. The y-axis is redefined as $\xi = x'_t(m)/v'_{t+1}(m)$, and, as discussed below, ξ is constant overtime as balanced growth. We will show that the equilibrium moves from intersection A to B, and the first-order derivative of the buyer's Bellman equation shifts upward due to the constant rate of decline in $v'_t(m)$ overtime.

First, Nash bargaining's condition of Figure 2 remains unchanged, because the Nash bargaining derives only the intratemporal condition. Specifically, in the non-stationary economy, the first-order condition (4) can be written as follows.

$$(1-\theta)c\left[k + x_t(m) + \beta\big(v_{t+1}(0) - v_{t+1}(m)\big)\right] = \theta\left[-d - cx_t(m) + \beta\big(v_{t+1}(m) - v_{t+1}(0)\big)\right], \quad (11)$$

which means that $\xi = x'_t(m)/v'_{t+1}(m)$ is unchanged from the stationary case (8). Furthermore, we can easily derive from (8) that $\xi > \beta$. Intuitively, by the marginal increase in m, the buyer gains $x'_t(m)$ and loses $\beta v'_{t+1}(m)$, while the seller loses $cx'_t(m)$ and gains $\beta v'_{t+1}(m)$.

In the Nash bargaining, the ratio between the buyer's and seller's surpluses is θ to $(1-\theta)c$, and this holds for all m. Since the seller's current loss $cx'(m)$ is smaller than the buyer's current gain $x'(m)$, to maintain this ratio, $x_t(m)$ increases more than $\beta v_{t+1}(m)$.

Next, we show the shift of the derivative of the buyer's Bellman equation in Figure 2 caused by a constant rate of decline in $v_t'(m)$. In the non-stationary economy, the Bellman equation (6) is rewritten as

$$v_t(m) = \alpha\tilde{H}(0)\big(k + x_t(m) + \beta v_{t+1}(0)\big) + \big(1-\alpha\tilde{H}(0)\big)\beta v_{t+1}(m) \quad \text{for } m \in [\underline{z}, \overline{Z}].$$

By taking the derivative with respect to m and dividing both sides by $v_{t+1}'(m)$, we obtain

$$\begin{aligned}\frac{v_t'(m)}{v_{t+1}'(m)} &= \alpha\tilde{H}(0)\left(\frac{x_t'(m)}{v_{t+1}'(m)}\right) + \big(1-\alpha\tilde{H}(0)\big)\beta \\ &= \alpha\tilde{H}(0)\xi + \big(1-\alpha\tilde{H}(0)\big)\beta. \end{aligned} \tag{12}$$

The left-hand side of (12) is the inverse of the growth rate of $v_t'(m)$ and it is one if $\tilde{H}(0) = H(0)$. When $\tilde{H}(0) > H(0)$, it is larger than one, because the right-hand side increases due to $\xi > \beta$. Then, $v_t'(m)$ non-stationary declines. That is, with an increase in $\tilde{H}(0)$, the current marginal value of m before matching, $v_t'(m)$, increases relative to the future one, $v_{t+1}'(m)$. Intuitively, on the right-hand side of (12), the increase in current marginal value $v_t'(m)$ due to the higher probability of buying, $\alpha\tilde{H}(0)x_t'(m)$, compensates for the decrease in marginal value due to the lower probability of non-matching, $\alpha\tilde{H}(0)\beta v_{t+1}'(m)$, because $x_t'(m) > \beta v_{t+1}'(m)$. Under $v_t'(m)/v_{t+1}'(m) > 1$ on the left-hand side of (12), considering ξ and $\tilde{H}(0)$ as variables, the downward sloping curve shifts to the dashed curve in Figure 2. In other words, the non-stationary decline in $v_t'(m)$ shifts the curve so that its intersection with the Nash bargaining's condition is at B. For $\tilde{H}(0) < H(0)$, equation (12) implies a constant positive growth of $v_t'(m)$. Note that the above discussion is limited to the local dynamics around a steady-state pay-all equilibrium, and globally, the pay-all property may eventually break down.

Macro-level variables. Given $H(0)$, macro-level variables are determined by a simple system of linear equations. First, we define the average discounted value of positive money holders as

$$\overline{v}_b = \frac{\int_{\underline{z}}^{\overline{Z}} v(m)dH(m)}{1 - H(0)}. \tag{13}$$

Next, the total output of the economy is

$$Y = \alpha H(0) \int_{\underline{z}}^{\overline{Z}} x(m)dH(m). \tag{14}$$

In (14), $x(m)$ is the amount of production in the meeting of a buyer with m and a seller without money. Hence, $\int_{\underline{z}}^{\overline{Z}} x(m)dH(m)$ would be the total production if all buyers succeeded in matching. We multiply this by meeting probability $\alpha H(0)$ to obtain the actual production. Then, the average production per meeting is

$$\overline{x} = \frac{Y}{\alpha H(0)(1 - H(0))} = \frac{\int_{\underline{z}}^{\overline{Z}} x(m)dH(m)}{1 - H(0)}. \tag{15}$$

By integrating over $m \in [\underline{z}, \overline{Z}]$, equations (5), (6), and (7) are rewritten as macro-level equations:

$$c\overline{x} = \beta[\theta + (1-\theta)c]\big(\overline{v}_b - v(0)\big) - [(1-\theta)ck + \theta d], \tag{16}$$

$$(1-\beta)\overline{v}_b = \alpha H(0)\left[k + \overline{x} + \beta\big(v(0) - \overline{v}_b\big)\right], \tag{17}$$

$$(1-\beta)v(0) = \alpha\big(1 - H(0)\big)\left[-d - c\overline{x} + \beta\big(\overline{v}_b - v(0)\big)\right]. \tag{18}$$

These equations imply our model's close relationship with the so-called second-generation models with indivisible money and Nash bargaining, such as Shi (1995) and Trejos and Wright (1995). Once $H(0)$ is given, the system of equations (16), (17), and (18) is parallel to the indivisible money models. Our $\overline{v}_b$ corresponds to the value of an agent holding one unit of money in these models and $\overline{x}$ is the amount of goods traded with one unit of money. In the indivisible money models, the total money supply is $1 - H(0)$, where $H(0)$ is the measure of zero money holders. In these models, each agent with one unit of money becomes a buyer with probability α and can trade with probability $H(0)$, whereas each agent without money becomes a seller with probability α and can trade with probability $1 - H(0)$. This meeting pattern is the same as the pattern in our model. Moreover, the quantity of goods

x is determined as an interior solution of Nash bargaining, which is similar to the pay-all equilibrium. Therefore, our macro-level equilibrium equations are as tractable as indivisible money models.

However, there is a fundamental difference in that $H(0)$ is determined endogenously and does not depend on the money supply M in our model. Therefore, in our model, there is room for policy intervention to improve social welfare by changing $H(0)$ even if M is fixed. Another innovation is the non-degenerate distribution of money holdings among potential buyers, while in Shi (1995) and Trejos and Wright (1995), it is degenerate and all money holders have one unit of money. Our model allows for redistributive policy exercises that adjust each agent's money holding. In indivisible money models, policies are limited to changing $H(0)$.

From equation (4), the ratio of the buyer's surplus to the seller's surplus is $\theta/[(1-\theta)c]$. This ratio is the same at the aggregate level because of the linearity of the preferences. Then, equations (17) and (18) imply that

$$\frac{\overline{v}_b}{v(0)} = \left(\frac{H(0)}{1-H(0)}\right)\left(\frac{\theta}{(1-\theta)c}\right). \tag{19}$$

The ratio of the average discounted value of the buyers to that of the sellers is determined as the surplus ratio multiplied by the population ratio. Similar properties can be found in indivisible money models.

Proposition 1. *In a pay-all equilibrium,*

$$\overline{v}_b = \frac{\theta(1-\beta)(ck-d)}{\beta(1-\theta)(1-c)[\alpha\beta\theta(1-c)-(1-\beta)c]}, \tag{20}$$

$$v(0) = \frac{ck-d}{\beta(1-c)}, \tag{21}$$

$$\overline{x} = \frac{\theta(1-\beta)(1-c)[\theta(ck-d)-cd]+c^2(k-d)(1-\beta)-\alpha\beta\theta(1-\theta)c(1-c)(k-d)}{(1-\theta)c(1-c)[\alpha\beta\theta(1-c)-(1-\beta)c]}. \tag{22}$$

Proof. The above equalities are obtained by solving the system of linear equations (16), (17), and (18), where $H(0)$ is given by (9). □

In equations (20) and (21), the discounted utilities are positive if $ck > d$ holds. Therefore, a sufficiently large $k > 0$ supports the individual rationality of market participation as discussed in Section 2.

This result also implies the determinacy of social welfare, W, which is defined as

$$W \equiv H(0)v(0) + \int_{\underline{z}}^{\overline{Z}} v(m)dH(m) = H(0)v(0) + (1 - H(0))\overline{v}_b. \tag{23}$$

Social welfare W is also expressed as the discounted sum of the total surplus multiplied by the number of meetings. By the linearities of the utility and cost functions,

$$W = \frac{\alpha H(0)\big(1 - H(0)\big)[(1 - c)\overline{x} + k - d]}{1 - \beta}. \tag{24}$$

From (8), (19), and (23), it can also be written as

$$W = \frac{H(0)v(0)\xi}{\beta(1 - \theta)}. \tag{25}$$

Micro-level indeterminacy. As shown above, macro-level variables $H(0)$, $\overline{v}_b$, $v(0)$, and $\overline{x}$ are uniquely pinned downed by the equilibrium conditions. By contrast, $H(m)$ for $m \in [\underline{z}, \overline{Z}]$ and the shape of the value function $v(m)$ for $m > 0$ are indeterminate. This is due to a lack of equations. There are two remaining equilibrium conditions: (1) and (2). There are also a few conditions for the existence of pay-all equilibria, as outlined in the next section. However, even if we consider these conditions, a continuum of functions v and H can be pay-all equilibrium value functions and the distribution of money holdings, respectively. As shown in the proof of the existence of the pay-all equilibrium, only inequality conditions on the shape of the value function are required. Then, micro-level indeterminacy occurs among positive money holders, owing to the indeterminacy of v and H for $m \in [\underline{z}, \overline{Z}]$. Furthermore, this indeterminacy is not *nominal* but *real* because $x(m)$ and $v(m)$ are indeterminate for $m \in [\underline{z}, \overline{Z}]$. Note that a perturbation eliminates the micro-level indeterminacy, that is, only one distribution survives under the perturbation. (See Section 3.2.)

Our result differs from other real indeterminacies found in some models of divisible money, such as Green and Zhou (1998). This type of indeterminacy arises due to some identity hidden in the monetary exchange.[6] It causes both macro- and micro-level indeterminacies and leads to the social welfare indeterminacy. Specifically, in these models, $H(0)$ is typically

[6]See Kamiya and Shimizu (2006) and Kamiya and Shimizu (2007) for the finite support of the distribution of money holdings and Kamiya (2019) for the infinite support.

indeterminate and the discounted values and social welfare depend on $H(0)$. In our pay-all equilibrium, this type of indeterminacy does not occur because $H(0)$ is uniquely determined.

3.2 A selection of distributions by a perturbation

Despite the micro-level indeterminacy above, we can demonstrate that the application of a perturbation method, specifically a trembling hand, effectively resolves this indeterminacy. Specifically, our analysis reveals that non-stationary distributions invariably converge towards a unique two-point distribution. This method of selection notably enhances model tractability and stability.

As in the original model, we assume that buyers have incentives for pay-all. However, due to a small tremble, each buyer cannot spend all the money holding; that is, at each bargaining, the buyer with m_b cannot pay ζm_b and uses only $(1-\zeta)m_b$ unit of money, where $\zeta > 0$ is a small number. Under this *pay-almost-all* property, there may exist a non-degenerate distribution $\tilde{H}_t$, which consists of positive measures in two ranges: $[\underline{z}, \overline{Z}]$ and $[\zeta\underline{z}, \zeta\overline{Z}]$. We hereafter add subscript t to consider non-stationary distributions. Since the same measure of agents are exchanged between two ranges, $\tilde{H}_t$ satisfies

$$\tilde{H}_t([\zeta\underline{z}, \zeta\overline{Z}]) = H(0) \quad \text{and} \quad \tilde{H}_t([\underline{z}, \overline{Z}]) = 1 - H(0), \tag{26}$$

where H is the stationary distribution in the original pay-all equilibrium.

We show that the unique stationary distribution is a two-point distribution where $\underline{z} = \overline{Z}$ and has two masses at

$$\tilde{m}_b \equiv \frac{M}{1 - H(0) + \zeta H(0)} \quad \text{and} \quad \tilde{m}_s \equiv \frac{\zeta M}{1 - H(0) + \zeta H(0)} = \zeta \tilde{m}_b. \tag{27}$$

Moreover, any non-degenerate distribution $\tilde{H}_t$ satisfying (26) is shown to converges to this two-point distribution (27).[7] To demonstrate this convergence, we represent the distribution using random variables. We define a set function

$$\tilde{H}_{s,t}(D) \equiv \frac{\tilde{H}_t(D)}{H(0)} \quad \text{for a Borel set} \quad D \subset [\zeta\underline{z}, \zeta\overline{Z}].$$

[7] Note that, under two-point distribution (27), the equilibrium condition is satisfied as in the original case. Suppose $\tilde{v}$ is the associated value function. Then, the equilibrium condition holds with $\tilde{v}(\tilde{m}_s) = v(0)$ and $\tilde{v}(\tilde{m}_b) = \overline{v}_b$. $H(0)$ is determined as in Lemma 1 by taking the derivative of $\tilde{v}$ in the neighborhood of $\tilde{m}_b$.

Then, $\tilde{H}_{s,t}$ is a probability measure of sellers. Similarly, we define the probability measure of buyers as

$$\tilde{H}_{b,t}(D) \equiv \frac{\tilde{H}_t(D)}{1 - H(0)} \quad \text{for a Borel set} \quad D \subset [\underline{z}, \overline{Z}].$$

Let $X_{s,t}$ and $X_{b,t}$ are random variables following $\tilde{H}_{s,t}$ and $\tilde{H}_{b,t}$, respectively. The means of both variables are constant with $\mathbb{E}[X_{s,t}] = \tilde{m}_s$ and $\mathbb{E}[X_{b,t}] = \tilde{m}_b$. However, variances $Var[X_{s,t}]$ and $Var[X_{b,t}]$ can be non-stationary. In the following proposition, we show that the variances converge to zero as $t \to \infty$, and therefore $X_{s,t}$ and $X_{b,t}$ converge stochastically to $\tilde{m}_b$ and $\tilde{m}_s$.

Proposition 2. *Suppose that both $\tilde{H}_{s,t}$ and $\tilde{H}_{b,t}$ are non-degenerate. Then, there exists a $\bar{\zeta} > 0$ such that for all $\zeta \in (0, \bar{\zeta})$, $Var[X_{s,t}] \to 0$ and $Var[X_{b,t}] \to 0$ as $t \to \infty$.*

Proof. See the Appendix. □

The intuitive reason for convergence is as follows. Consider agents (buyers) with $m > \tilde{m}_b$. Trading partners (sellers) will obtain $(1-\zeta)m$ in the trade, whereas their money holding is m_s with mean being $\zeta\tilde{m}_b$. That is, the ζ fraction of the money holding is replaced with m_s, with the mean being $\zeta\tilde{m}_b$. Therefore, the total money holdings of agents with $m > \tilde{m}_b$ approaches $\tilde{m}_b$. A similar argument applies to agents (buyers) with $m < \tilde{m}_b$. For agents (sellers) with $m \in [\zeta\underline{z}, \zeta\overline{Z}]$, trading partners' money holdings approach $\tilde{m}_b$ as discussed above. Therefore, their money holdings converge to $\zeta\tilde{m}_b$.

4 Existence

We now show the existence of a continuum of pay-all equilibria using a guess-and-verify method. Namely, we present a candidate for the pay-all equilibrium and prove that it satisfies the equilibrium conditions. As stated in Section 3, on-path behavior satisfies the pay-all property between a seller holding $m_s = 0$ and a buyer holding $m_b \in [\underline{z}, \overline{Z}]$. Moreover, we consider off-path behaviors, that is, the case of agents holding $m \notin \{0\} \cup [\underline{z}, \overline{Z}]$.

Each agent's choice of trade depends on a cut-off level of money holding m^1. On the off-path, an agent holding $m < m^1$ may sell goods but never purchases whoever this agent meets. However, an agent holding $m \geq m^1$ does not sell goods at any meeting, but purchases goods when she is a buyer and the partner has $m_s \in [0, m^1)$. We show that this cut-off strategy is optimal and holds at equilibrium. The equilibrium is verified by checking the no-deviation of each agent, given that all the others follow the pay-all equilibrium under the stationary equilibrium distribution of money holdings. Since the measure of this agent is zero, we exclude strategic reactions to each deviation. Here, we use $x(m_b, m_s, H)$ and $p(m_b, m_s, H)$ instead of $x(m)$ and $p(m)$ because we consider off-path cases. The associated Bellman equations are

$$v(m) = \alpha H(0)\big(k + x(m, 0, H) + \beta v(0)\big) + \big(1 - \alpha H(0)\big)\beta v(m) \quad \text{for } m \in [m^1, \overline{m}], \tag{28}$$

$$v(m) = \alpha \int_{\underline{z}}^{\overline{Z}} \big(-d - cx(m_b, m, H) + \beta v(m + m_b)\big) dH(m_b) + \left[1 - \alpha\big(1 - H(0)\big)\right] \beta v(m) \quad \text{for } m \in [0, m^1). \tag{29}$$

Equation (28) is a generalization of (6), and (29) is a generalization of (7). Since $[m^1, \overline{m}]$ is an expanded range for being a buyer, it must include on-path cases. Therefore,

$$m^1 \;\leq\; \underline{z} \;\leq\; \frac{M}{1 - H(0)} \;\leq\; \overline{Z} \;\leq\; \overline{m}. \tag{30}$$

These Bellman equations are associated with the following cases of meeting and trade. We consider both the on-path and off-path behaviors of a single agent given that all the others follow the pay-all equilibrium with the stationary money holding distribution. This agent holds $m \in \mathbb{R}_+$ and matches with other agents holding $m \in \{0\} \cup [\underline{z}, \overline{Z}]$. Then, in this economy, the following four on-path and off-path bargaining outcomes are expected in the pay-all equilibria at the beginning of each period (see Figure 3 for a graphical representation).

Equilibrium Bargaining Outcomes (EBOs)

1. Among the on-path cases, a trade occurs and the pay-all property holds between
   - $m_s = 0$ and $m_b \in [\underline{z}, \overline{Z}]$
2. Trades never occur in the other on-path cases:
   - $m_s = m_b = 0$
   - $m_s \in [\underline{z}, \overline{Z}]$ and $m_b = 0$

Figure 3: Equilibrium Bargaining Outcomes (EBOs)

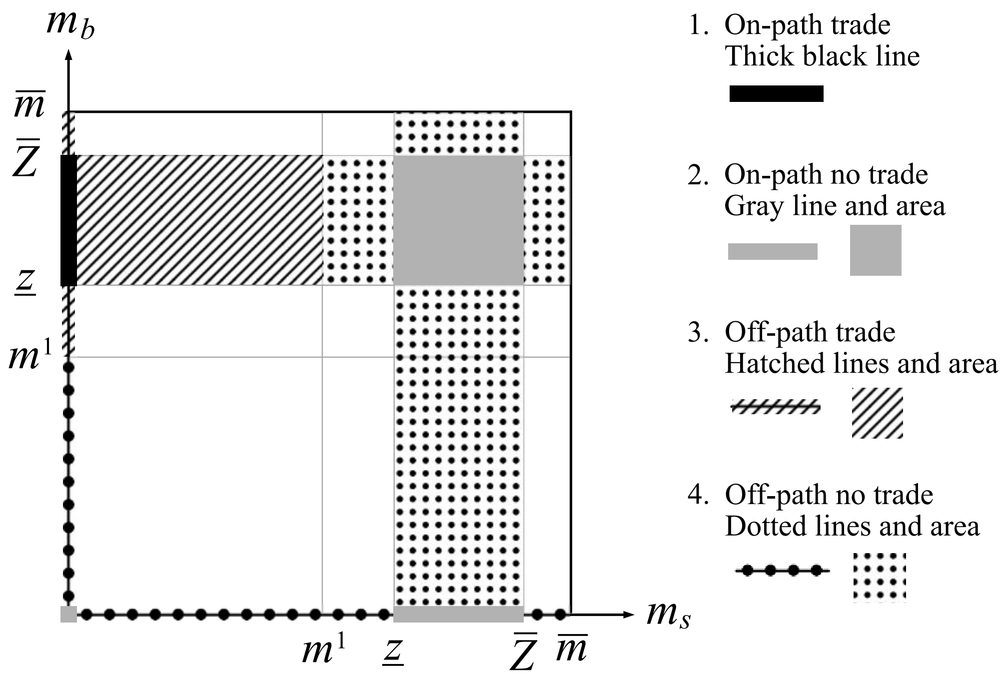
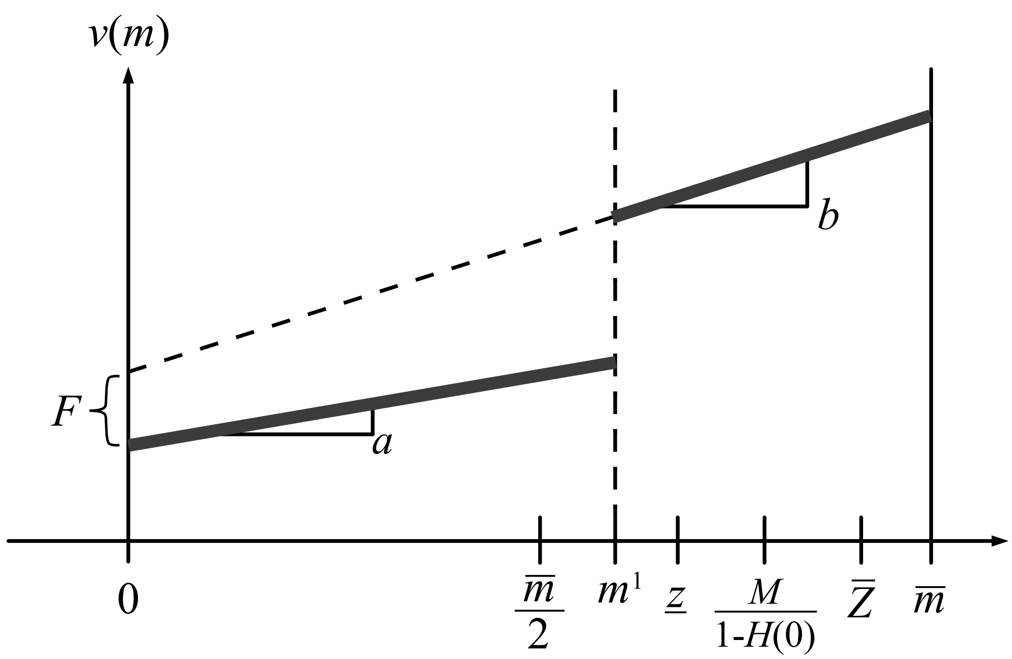


Figure 4: Equilibrium value function



- $m_s \in [\underline{z}, \overline{Z}]$ and $m_b \in [\underline{z}, \overline{Z}]$

3. Among the off-path cases, a trade occurs and the pay-all property holds between

   - $m_b \in [\underline{z}, \overline{Z}]$ and $m_s \in (0, m^1)$
   - $m_s = 0$ and $m_b \in [m^1, \overline{m}] \setminus [\underline{z}, \overline{Z}]$

4. Trades never occur in the other off-path cases:

   - $m_b = 0$ and $m_s \in (0, m^1)$,
   - $m_b \in \{0\} \cup [\underline{z}, \overline{Z}]$ and $m_s \in [m^1, \overline{m}] \setminus [\underline{z}, \overline{Z}]$
   - $m_s \in \{0\} \cup [\underline{z}, \overline{Z}]$ and $m_b \in (0, m^1)$
   - $m_s \in [\underline{z}, \overline{Z}]$ and $m_b \in [m^1, \overline{m}] \setminus [\underline{z}, \overline{Z}]$

Finally, the following is a candidate equilibrium value function that satisfies (28) and (29).

$$v(m) = \begin{cases} \frac{ck-d}{\beta(1-c)} + am & \text{for } m \in [0, m^1), \\ \frac{ck-d}{\beta(1-c)} + F + bm & \text{for } m \in [m^1, \overline{m}], \end{cases} \tag{31}$$

$$\text{where } b > a > 0. \tag{32}$$

The value function is shown in Figure 4. Here, m^1, a, b, and F are endogenous variables. At the cut-off level of buyer/seller choice m^1, the value function has a jump associated with gap F. The slopes are represented by a and b. The value function is indeterminate, that is, m^1, a, b, and F can take any number within a certain range consistent with no deviation condition from the EBOs.

The shape of the candidate value function is designed to satisfy the on-path equilibrium behavior and eliminate off-path deviations as follows. The linearity of the value function derives the pay-all property. If $v(m)$ is concave, buyers may save a small amount of money for future purchases. However, under the linearity assumption, the marginal value of money savings is always bounded by $\alpha H(0)\beta b$. If it is sufficiently small, buyers have incentives for pay-all.

One role of the jump at m^1 is to derive the on-path behavior. After each trade, the seller's discounted value increases from $v(0)$ to $v(m)$ for $m \in [\underline{z}, \overline{Z}]$. The jump in $v(m)$ at

m^1 covers fixed costs d. This jump is necessary to maintain the buyer's incentive to pay-all. If there is no jump, to induce the seller's incentive to sell goods, the slope of $v(m)$ should be sufficiently steep. However, this would incentivize the buyers to save money.

Another role of the jump is to eliminate off-path trades when both the buyer and seller hold $m \geq m^1$. If a trade occurs, the seller's discounted value would not jump because $m_s \geq m^1$. Then, the increase in $v(m)$ would be too small to cover fixed costs d and, thus, the bargaining would fail. There is another off-path case that we need to exclude: the seller's money holding m_s satisfies $0 < m_s < m^1$ and $m^1 - m_s < m_b - m^1$, where m_b is the buyer's money holding. If the buyer pays $m_b - m^1$, then the increase in the seller's value includes the jump at m^1, but the decrease in the buyer's value does not include the drop at m^1. They are likely to reach an agreement and deviate from EBOs. To eliminate this case, we need the condition that the gain from obtaining $m_b - m^1$ is not sufficiently large to cover fixed costs.

Before presenting an existence theorem, we explain the reason the theorem of maximum fails and the value function is discontinuous. In Green and Zhou (1998)'s indivisible goods model, the theorem of maximum fails because the value function is a (discontinuous) step function due to a jump in the cost function. That is, the cost is zero for no-production and a positive number for the production of indivisible goods. In our model, the reason why the theorem of maximum fails is similar to that in Green and Zhou (1998): the objective function is discontinuous at zero due to fixed costs. Another reason may be that it is not a single-person optimization but a game-theoretic optimization. In our model, a buyer and a seller jointly maximize the Nash product, and thus the theorem of maximum cannot be directly applicable.

Proposition 3. *Suppose that the following conditions are satisfied.*

$$H(0) = \frac{(1-\beta)c}{\alpha\beta\theta(1-c)} < 1, \tag{33}$$

$$v(0) = \frac{ck - d}{\beta(1-c)} > 0, \tag{34}$$

$$\frac{\overline{m}}{2} < \frac{M}{1 - H(0)} < \overline{m}, \tag{35}$$

Moreover, given $A \equiv \frac{(1-\theta)ck+\theta d}{\beta[(1-\theta)c+\theta]}$, *if either*

$$\frac{A}{1-c} < \overline{v}_b - v(0) \leq \frac{d(1-c)}{c} \tag{36}$$

or

$$\max\left[\frac{Ac}{(1-c)(\overline{v}_b - v(0) - A)}, \left(\frac{M}{1-H(0)}\right)^{-1}\frac{\overline{m}}{2}\right] < \min\left[\frac{d}{\overline{v}_b - v(0) - \frac{d(1-c)}{c}}, 1\right] \tag{37}$$

holds, then there exists a continuum of stationary pay-all equilibria.

Proof. See the Appendix. □

Note that some parameters satisfy all conditions in Proposition 3.[8] The intuition of the sufficient conditions is as follows. Inequality (33) ensures that the population of zero-money holders is lower than the total population, while inequality (34) corresponds to the individual rationality of participating in the search market.

In inequality (35), $M/(1-H(0))$ is the average money holding. The first inequality means that the money holdings are close enough to the upper-bound, $\overline{m}$. Then, the agent holding $m > 0$ declines to be a seller because the monetary profit is limited by $\overline{m} - m$. The second inequality simply means that the average money holding does not exceed the upper bound.

Inequalities (36) and (37) are incentive conditions that prevent deviation from EBOs. For a clear intuition, we focus on (36). The first inequality means that, since the seller's gain from trade $\overline{v}_b - v(0)$ is sufficiently large, a positive amount of goods is traded as the interior solution of Nash bargaining problems. By (16), this inequality is sufficient for $\overline{x} > 0$. The second inequality excludes an off-path seller's incentive to produce goods for poor buyers. Specifically, fixed costs d must be sufficiently large relative to the equilibrium seller's gain from trade $\overline{v}_b - v(0)$. Moreover, c should be sufficiently small to make fixed costs d relatively more important and strengthen the above incentive.

The condition in Proposition 3 is sufficient, and pay-all equilibrium may exist even if this is not met. In some cases, however, pay-all equilibrium clearly does not exist. For example,

[8] For example, our benchmark parameters are $\alpha = 0.1$, $\beta = 0.9$, $c = 0.17$, $k = 1.0$, $d = 0.1$, $\theta = 0.5$, $M = 1$, and $\overline{m} = 3$. These meet (33), (34),(35), and (36) with strict inequalities. Alternatively, if c is changed to 0.18, (36) is violated, but (37) is satisfied.

when c or k is close to zero, then from Proposition 1, $v(0)$ and $\overline{v}_b$ are negative. Another example is when θ is close to 0. In this case, from equation (22), $\overline{x}$ is negative and there does not exist a pay-all equilibrium. The last example is when d is close to zero. Then sellers will sell goods even to poor buyers, that is, the off-path EBOs are not satisfied.

Finally, we discuss the case of a large $\overline{m}$ that does not satisfy the first inequality in (35). Then a pay-all equilibrium may no longer exist because a seller with $m_s \in [\underline{z}, \overline{Z}]$ may sell goods to a buyer with a large amount of money, and the off-path EBOs may not be satisfied.

5 Bargaining power and social welfare

The Hosios condition, a condition on bargaining power θ for efficiency, is often discussed in the literature on search models. In our model, the impact of θ on social welfare W is non-monotone and simply characterized by the following proposition.[9]

Let the average surpluses of buyers and sellers be $B = k + \overline{x} + \beta(v(0) - \overline{v}_b)$ and $S = -d - c\overline{x} + \beta(\overline{v}_b - v(0))$, respectively. From (24), social welfare is written as

$$W = \frac{\alpha H(0)(1 - H(0))}{1 - \beta}(B + S).$$

From (4), $B = \big(\theta/[(1-\theta)c]\big)S$, and from (18), $S = [(1-\beta)v(0)]/[\alpha(1 - H(0))]$. Therefore,

$$B + S = \left(\frac{\theta}{(1-\theta)c} + 1\right) S = \left(\frac{\theta}{(1-\theta)c} + 1\right) \frac{(1-\beta)v(0)}{\alpha(1 - H(0))},$$

and by $H(0) = (1-\beta)c/(\alpha\beta\theta(1-c))$,

$$W = \left(\frac{\alpha H(0)(1 - H(0))}{1-\beta}\right)\left(\frac{\theta}{(1-\theta)c} + 1\right)\left(\frac{(1-\beta)v(0)}{\alpha(1 - H(0))}\right) \tag{38}$$

$$= \left(\frac{(1-\beta)cv(0)}{\alpha\beta(1-c)}\right)\left(\frac{1}{\theta}\right)\left(\frac{\theta}{(1-\theta)c} + 1\right). \tag{39}$$

In (38), W is decomposed into three elements: the first parenthesis represents the number of matchings multiplied by $(1-\beta)^{-1}$, the second represents the ratio of B to S, $\theta/(1-\theta)c$, plus one, and the third represents S.

[9] For simplicity, we assume that a pay-all equilibrium exists even if θ is changed. A large change in θ may violate the sufficient conditions for existence in Proposition 3.

Proposition 4. *The sign of the welfare improvement is determined as*

$$\frac{\partial W}{\partial \theta} > 0 \text{ if and only if } \theta > \frac{c}{c + \sqrt{c}}. \tag{40}$$

Proof. The derivative of $\left(\frac{1}{\theta}\right)\left(\frac{\theta}{(1-\theta)c} + 1\right)$ in (39) with respect to θ is

$$\left(\frac{\theta}{(1-\theta)c} + 1\right)\frac{-1}{\theta^2} + \frac{(1-\theta)c + c\theta}{(1-\theta)^2 c^2}\frac{1}{\theta} = \frac{1}{c\theta^2(1-\theta)^2}((1-c)\theta^2 + 2c\theta - c).$$

Since $(1-c)\theta^2 + 2c\theta - c = 0$ has one negative and one positive solution, $\theta = \frac{c}{c-\sqrt{c}}$ and $\theta = \frac{c}{c+\sqrt{c}}$, the proposition holds. □

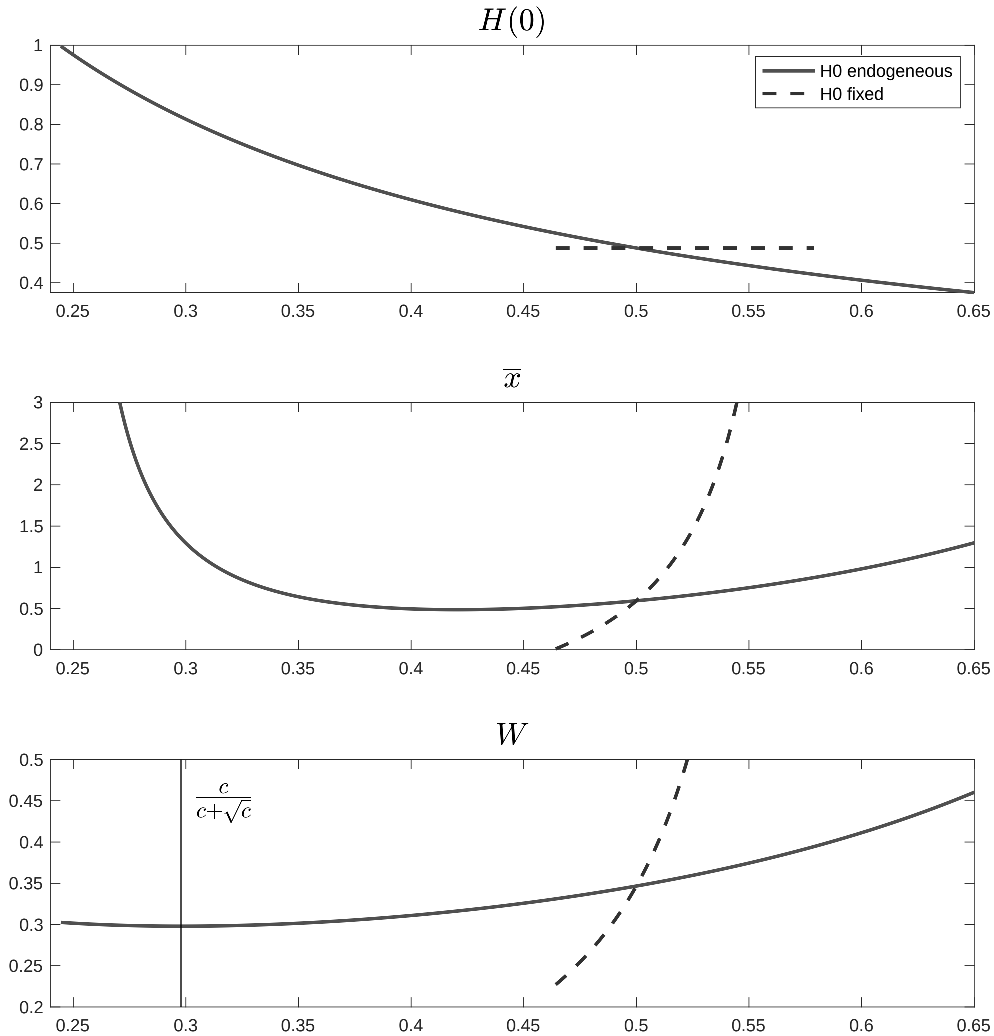
To interpret the above proposition, we analyze the effect of θ on $H(0)$, $\overline{x}$, and W. Figure 5 illustrates the comparative statistics for $H(0)$, $\overline{x}$, and W with respect to θ. The solid line represents the baseline results and the dashed line represents the case with fixed $H(0)$. Condition (40) corresponds to the social welfare's U-shaped relationship of the solid line in the bottom diagram. This nonlinearity of social welfare is caused by a combination of three separate effects.

1. Number of matchings: The first parenthesis of (38) means that more matchings increase social welfare given the average production is unchanged. Since the number of matchings is $\alpha H(0)\big(1 - H(0)\big)$, $H(0) = 1/2$ is the best.

2. Change in the bargaining power θ on $\overline{x}$: In the middle panel of Figure 5, the dashed line is increasing in θ. As buyers gain more bargaining power, matched sellers are required to produce more. This is the direct effect in the bargaining problem (3). Moreover, this effect is magnified by an increase in buyers' discounted utility, $v(m)$. Each buyer's threat point of Nash bargaining $\beta v(m)$ rises and it leads to an increase in the matched seller's production. Since the marginal utility, 1, is always larger than variable costs, c, social welfare increases as the average production $\overline{x}$ increases. These effects explain the increasing part of social welfare: the solid line in the bottom figure. In our model, these bargaining power channels are represented in the second parenthesis of (38). As θ increases, the ratio of B to S, $\theta/(1-\theta)c$, also rises.

3. Change in $H(0)$ on $\overline{x}$: This channel corresponds to the decreasing part of the solid line in the middle panel. A decline in $H(0)$ implies that the measure of sellers becomes

Figure 5: Comparative statics with respect to θ



Notes: The solid line represents the results of the baseline model and the dashed line the case with fixed $H(0)$. In this exercise, all parameters except for θ are the same as the benchmark parameters in Footnote 8. The fixed $H(0)$ is calculated given $\theta = 1/2$. Although the solid line shows a U-shaped relationship in the bottom diagram, the decreasing region is quantitatively limited. This is due to the constraint of $H(0) < 1$ for θ being lower than 0.25. The range of dashed lines is also restricted for θ between 0.46 and 0.58. This is because $\overline{x}$ becomes negative if θ is too low and $\overline{x}$ does not exist for a too high θ. Social welfare W is minimized at $\theta = c/(c + \sqrt{c})$, as proven by Proposition 4.

relatively smaller than that of buyers. It limits the buyer's chance of trade and reduces the average discounted value of buyers, $\overline{v}_b$, relative to the value of sellers $v(0)$. Then, being buyers becomes less attractive for sellers. It decreases $\overline{x}$ and worsens the holdup problem. In (38), this effect is represented by the third parenthesis that equals to S.

Given that $v(0)$ is constant, a decline in $H(0)$ reduces the future value of being a buyer, which is included in S. This effect is multiplied by the second parenthesis and also changes B.

For a low θ, the middle panel implies that the third channel dominates the second. Since $H(0)$ is close to 1, a decline in $H(0)$ drastically drops down the ratio between the discounted utilities of buyers and sellers following equation (19). However, W is nearly flat because, again, by $H(0)$ close to 1, the number of matchings increases significantly in the first channel and offsets the second. For a high θ, $H(0)$ is still in the middle range. Even if $\theta = 1$, $H(0) > 0$ from equation (9). Then, the first and third channels are mild, and the second channel increases W.

Condition (40) depends only on c because the three channels cancel each other out in equation (38). First, the third parenthesis represents the relative value of the buyer to the seller, which depends inversely on $1 - H(0)$. Next, this term cancels out the matching probability's $1 - H(0)$ in the first parenthesis. Then, θ remains in the second parenthesis and is included in $H(0)$ in the first parenthesis. Finally, by the linearities of the utility and cost functions, all factors, including parameters other than θ and c, become multiplicative as in (39) in the proof.

Our model's U-shape result is unique in the literature because it is caused by a change in $H(0)$. If $H(0)$ is fixed, only the second channel works as the dashed lines and social welfare increases monotonically. This case is similar to the indivisible money models (Shi, 1995; Trejos and Wright, 1995), where the distribution is fixed by the money supply as $1 - H(0) = M$. This monotonicity also appears in Lagos and Wright (2005) because there are no distributional effects.

6 Redistributive monetary transfer under the budget balance

We here consider a long-run redistributional monetary transfer under fixed money supply and balanced government budget. This policy is implemented using a combination of per-unit tax (subsidy) and an asset tax (subsidy) at the steady state. We show that this policy changes the relative strengths in bargaining, defined below as effective bargaining power and can improve social welfare, as in Proposition 4.

At time 0, the government commits two time-independent taxes implemented after each trade under keeping the stationary equilibrium. First, the government collects (provides) a linear per-unit tax (subsidy) tx, where t is the tax (subsidy) rate if $t > 0$ ($t < 0$) and x is the amount of goods sold. Second, the government imposes an asset tax (subsidy) depending on the seller's gross money holdings. This tax depends on m, which is the amount of money after trade and before per-unit tax. The transfer schedule follows a function $g(m)$, which is a tax (subsidy) if $g(m) > 0$ ($g(m) < 0$).

Since both t and $g(m)$ are given to each agent, Nash bargaining is solved given the seller's after-tax money holdings

$$m - tx - g(m). \tag{41}$$

The government perfectly predicts the equilibrium outcomes at time 0, where production is represented as $x\big(m \mid t, g(m)\big)$. The government first decides t. Then, it can decide $g(m)$ to balance its budget for each *individual* m as the functional solution of

$$g(m) = -tx\big(m \mid t, g(m)\big). \tag{42}$$

See Appendix A.3 for the existence and derivation of $g(m)$. In contrast to the usual aggregate level budget balance, this policy is more strict and imposes the balance in micro-level. The distribution of money holdings remains unchanged over time. However, this policy matters for real variables because it separately affects the decision in production x and payment m for each Nash bargaining. Note that this policy implicitly assumes the perfect ability of the government to collect taxes. Although it appears inconsistent with the anonymity of the search market, in our interpretation, it can be implemented by government agents, as in Aiyagari and Wallace (1997).[10]

As in the baseline no-policy model, the system is characterized by three equations: the first-order condition of Nash bargaining problem, value function for positive money holders,

[10]There are many government agents who randomly meet pairs of private agents. They collect taxes (provide subsidies) and adjust the money holdings of the matched pairs. In this case, the policy applies to a subset of matched pairs. We implicitly assume that government agents find all matched pairs, whose measure is $\alpha H(0)(1 - H(0))$ for simplicity. Note that all arguments in this section remain valid with minor changes even when they match a subset of matched pairs, that is, the case that the government agents match $\pi\alpha H(0)(1 - H(0))$ measure of pairs, where $0 < \pi < 1$ is the meeting probability of each government agent.

and that for zero-money holders. For simplicity, we use $x(m)$ instead of $x\big(m \mid t, g(m)\big)$. First, the Nash bargaining problem is written as

$$\max_x \left[k + x + \beta\big(v(0) - v(m)\big)\right]^\theta \left[-d - cx + \beta\big(v(m - tx - g(m)) - v(0)\big)\right]^{1-\theta} \tag{43}$$

in a pay-all equilibrium. Then, under the budget balance (41), the first-order condition is

$$(1-\theta)\big(c + \beta t v'(m)\big)\left[k + x + \beta\big(v(0) - v(m)\big)\right] = \theta\left[-d - cx + \beta\big((v(m) - v(0)\big)\right]. \tag{44}$$

Next, the Bellman equation for a buyer is unchanged from equation (6). We write it again here for convenience.

$$v(m) = \alpha H(0)[k + x(m) + \beta v(0)] + [1 - \alpha H(0)]\beta v(m). \tag{6}$$

Finally, the seller's value function is also unchanged because

$$\begin{aligned} v(0) &= \alpha \int_{\underline{z}}^{\overline{Z}} \big(-d - cx(m) + \beta v(m - tx - g(m))\big) dH(m) + \left[1 - \alpha\big(1 - H(0)\big)\right]\beta v(0). \\ &= \alpha \int_{\underline{z}}^{\overline{Z}} \big(-d - cx(m) + \beta v(m)\big) dH(m) + \left[1 - \alpha\big(1 - H(0)\big)\right]\beta v(0). \end{aligned} \tag{7}$$

We suppose that t and $g(m)$ are sufficiently small to satisfy the pay-all property and inequalities in Proposition 3, which guarantees the existence of equilibria.

Below, we assume a linear value function, that is, slope $v' = v'(m)$ is a constant for $m \in [\underline{z}, \overline{Z}]$. As proven by Proposition 3, we can consider such a linear value function. First, we derive the population of zero-money holders.

Lemma 2.

$$H(0) = \frac{(1-\beta)[c + (1-\theta)\beta t v']}{\alpha\beta\theta(1-c)} \tag{45}$$

Proof. Equation (44) holds for all $m \in [\underline{z}, \overline{Z}]$ and x depends on m. By the first-order derivative with respect to m,

$$(1-\theta)(c + \beta t v')(x' - \beta v') = \theta(-cx' + \beta v').$$

As in the no-policy case, define $\xi(m) = x'(m)/v'(m)$. Then,

$$\xi = \beta + \frac{\beta\theta(1-c)}{c + \beta(1-\theta)tv'}.$$

Since (6) is unchanged, the buyer's condition of $\xi(m)$ is also unchanged from (10). Then, $H(0)$ is obtained from these two equations. □

Note that $H(0)$ depends on the per-unit tax rate t and the slope of the value function v'. Given the indeterminacy of the value function, $H(0)$ also becomes indeterminate.

To obtain an analytical solution, we convert the micro-level equations to macro-level equations using the average buyer's value $\overline{v}_b$ defined by (13) and average production $\overline{x}$ defined by (15). Under the linearity of $v(m)$, by the integration of (44) over $m \in [\underline{z}, \overline{Z}]$, we derive

$$(1-\theta)\left[c + \beta tv'\right]\left[k + \overline{x} + \beta(v(0) - \overline{v}_b)\right] = \theta\left[-d - c\overline{x} + \beta(\overline{v}_b - v(0))\right]. \tag{46}$$

To compare it with the no-policy case, we define the effective bargaining power as

$$\theta_e = \frac{\theta c}{\theta c + (1-\theta)(c + \beta tv')}. \tag{47}$$

Then, (46) is rewritten as

$$(1-\theta_e)c\left[k + x + \beta\big(v(0) - \overline{v}_b\big)\right] = \theta_e\left[-d - c\overline{x} + \beta\big(\overline{v}_b - v(0)\big)\right]. \tag{48}$$

This condition is the same as the equation of no-policy case (4) except that θ is replaced by θ_e in (48). In the case of no policy, $t = 0$ and $\theta_e = \theta$ hold. Next, we rewrite the population of zero money holders, (45), using θ_e as.

$$H(0; \theta_e) = \frac{(1-\beta)c}{\alpha\beta(1-c)\theta_e}. \tag{49}$$

Equations (47) and (49) provide another interpretation of the policy. By choosing the per-unit tax rate t (and implicitly $g(m)$), the government can directly control effective bargaining power θ_e, which also affects the distribution of money holdings.

Given θ_e and $H(0;\theta_e)$, the macro-level variables are obtained from (6), (7), and (48) as.

$$c\overline{x} = \beta[\theta_e + (1-\theta_e)c][\overline{v}_b - v(0)] - [(1-\theta_e)ck + \theta_e d]\,, \tag{50}$$

$$(1-\beta)\overline{v}_b = \alpha H(0;\theta_e)\{k + \overline{x} + \beta[v(0) - \overline{v}_b]\}, \tag{51}$$

$$(1-\beta)v(0) = \alpha[1 - H(0;\theta_e)]\{-d - c\overline{x} + \beta[\overline{v}_b - v(0)]\}. \tag{52}$$

This system of equations is the same as that in the no-policy case, except that θ is replaced with θ_e. The welfare effects of the policy are also equivalent to the comparative statics according to θ in Proposition 4. Let $W_g(\theta_e)$ be social welfare, given θ_e. The government policy changing θ_e coincides with the comparative statics for θ in the original case.

Proposition 5. *The sign of the welfare improvement of the policy is determined as*

$$\frac{\partial W_g(\theta_e)}{\partial \theta_e} > 0 \quad \textit{if and only if} \quad \theta_e > \frac{c}{c + \sqrt{c}}, \tag{53}$$

or equivalently

$$\frac{\partial W_g\big(\theta_e(t)\big)}{\partial t} > 0 \quad \textit{if and only if} \quad tv' > \frac{\theta\sqrt{c} - (1-\theta)c}{(1-\theta)\beta} \tag{54}$$

Proof. The system of equations, (50), (51), (52), and (49) is equivalent to (16), (17), (18), and (9). Therefore, $\frac{\partial W}{\partial \theta} = \frac{\partial W_g(\theta_e)}{\partial \theta_e}$ and the same conditions as in Proposition 4 hold. From equation (47), it is rewritten as

$$\theta_e(t) = \frac{\theta c}{\theta c + (1-\theta)(c + \beta t v')} > \frac{c}{c + \sqrt{c}}.$$

Since $\theta_e(t)$ is decreasing in t, (54) follows. □

Condition (53) implies that the marginal introduction of the policy to the laissez-faire economy derives the same welfare effects as the comparative statics on θ. This is the case of $\frac{\partial W_g(\theta_e)}{\partial \theta_e}$ evaluated at $\theta_e = \theta$ or, equivalently, at $t = 0$. Moreover, condition (54) implies that welfare $W_g\big(\theta_e(t)\big)$ is a U-shaped function, which is similar to Figure 5. Therefore, to improve social welfare, the government should make t as low as possible, or as large as possible, under consistency with the existence of the pay-all equilibrium.

Micro indeterminacy turns to macro. In the no-policy case, although the micro-level indeterminacy remains, the macro-level variables and social welfare are determinate. However, given $t \neq 0$, the micro-level indeterminacy about the slope of value function v' causes indeterminacy in $H(0)$ and W. This indeterminacy still remains even if the distribution of money holdings is selected by the tremble in Section 3.2.

Can the government achieve target allocation under this indeterminacy? If people's expectations about v' are fixed, the government can adjust t and perfectly control θ_e using (47). However, v' may change after policy implementation. The government can still determinate the allocation by introducing one more policy measure. For example, at the same time as offering t, the government can also adjust $H(0)$ by directly redistributing money between positive and non-money holders. Then, θ_e is uniquely chosen by (49) and v' is determined by (47).

Even if the equilibrium is unpredictable, this policy is still worth considering. Policy uncertainty is only about its magnitude. Equation (47) implies that, given the sign of t, the direction of the change in θ_e is predictable. Hence, given the U-shaped welfare function, the government can better steer the economy.

Necessity of per-unit tax. The per-unit tax is a crucial assumption in our exercise. It is independent from the monetary unit and this feature makes a crucial difference to fiscal and monetary policy in the literature depending on monetary transfer. For example, consider a combination of lump sum transfer and interest on money holdings to the seller.[11] The Nash bargaining problem can be written as

$$\max_x \left[k + x + \beta\big(v(0) - v(m)\big)\right]^\theta \left\{-d - cx + \beta\left[v\big((1+i_m)m + \overline{\tau}\big) - v(0)\right]\right\}^{1-\theta}, \tag{55}$$

where i_m is the interest on money and $\overline{\tau}$ is the lump-sum transfer. The government budget can be balanced with $i_m > 0$ and $\overline{\tau} < 0$. Under the Lagos-Wright framework, social welfare is maximized under the Friedman rule: $i_m = \beta^{-1} - 1$. However, equation (55) implies that this policy does not affect the first-order condition of the Nash bargaining with respect to x.[12] In the Lagos-Wright model, the interest rate affects marginal costs of saving money

[11] See, for example, Chapter 6.2 of Rocheteau and Nosal (2017)

[12] There may exist an indirect effects on x through the change in the distribution of money holdings. Given $i_m > 0$ and $\overline{\tau} < 0$, the distribution of positive money holders expands. Since this expansion is proportional and the average money holding remains unchanged, there is no effect on average production $\overline{x}$ and social

to the next period in the centralized market. It also eventually affects the intra-temporal condition in the decentralized market. However, in our model, this type of saving choice is inelastic to the interest rate because of the corner solution under the pay-all equilibrium. Therefore, to modify the intra-temporal condition in Nash bargaining, a policy tool directly affecting production, such as the per-unit tax, is necessary. This feature also implies the ineffectiveness of monetary transfers among non-matched agents in the stationary equilibrium because they conduct no production.[13] From this viewpoint, our policy explores another aspect of government intervention on the Friedman rule based on the literature.

7 Monetary Expansion

Here, we introduce monetary expansion to the model and analyze its welfare consequences. We obtain analytical results in a variety of lump-sum transfers and emphasize that monetary expansion is possibly effective when it is distributional.

The common framework is as follows. Let $\tau_t(m)$ be the monetary transfer for agent m in period t. The timing of the transfer is at the beginning of each period t. Suppose the money growth rate is μ_t. Then, the transfer $\tau_t(m)$ satisfies

$$\int_0^{\overline{m}_t} \tau_t(m) dH_t(m) = \mu_t M_t. \tag{56}$$

We assume that the upper bound of money holdings also grows with the same rate: $\overline{m}_{t+1} = (1+\mu_t)\overline{m}_t$. In the case of a proportional transfer, $\tau_t(m) = \mu m$ for all $m \in [0, \overline{m}_t]$, both constant and one-time injections cause no real effects, as in Molico (2006) for the former scenario.

Lump-sum transfer. First, we consider the lump-sum transfer with constant money growth: $\tau_t(m) = \mu M_t$ for all $m \in [0, \overline{m}_t]$. However, this lump-sum transfer requires some advancements for equilibrium analysis. Under the pay-all equilibrium, there exists a mass at $m = 0$. The no money holders accumulate the lump-sum transfers each period until successful matching with buyers. Then a non-degenerate distribution emerges from $m = 0$ and

welfare. However, the distribution of $x(m)$ possibly changes. This indirect effect is eliminated under the two-point distribution selected by a trembling hand in Subsection 3.2.

[13] In the non-stationary equilibrium, short-run policies changing the distribution of money holdings may matter. We provide one example of one-time lump-sum transfer with tax collection in Section 7.

expands toward the right. Eventually, this left-hand side distribution overlaps with the right-hand side distribution of $m \in [\underline{z}, \overline{Z}]$. The analytical characterization of this model requires another condition that uniquely determines cutoff point m^1, although it is indeterminate in the original model. See Appendix A.6 for a simulation result.

Another scenario is the one-time lump-sum transfer, that is, a helicopter drop. The money growth has been zero and expected to be zero for all future periods. However, only in one period, unexpectedly, each agent receives the fixed amount of the transfer. This case is also challenging because the right tail of the distribution extends toward infinity, and is discussed in Appendix A.6 in detail.

Lump-sum transfer with tax collection We provide an analytically solvable case of one-time lump-sum monetary injection with future tax collection. The government suddenly conducts a helicopter drop by distributing the fixed amount of money in period t. It announces that, in period $t+1$, it will collect money from positive amount holders. This combination of positive and negative transfers retain the tractability of the model because it makes the distribution of money holdings stationary after $t+1$. We also show that social welfare can be improved.

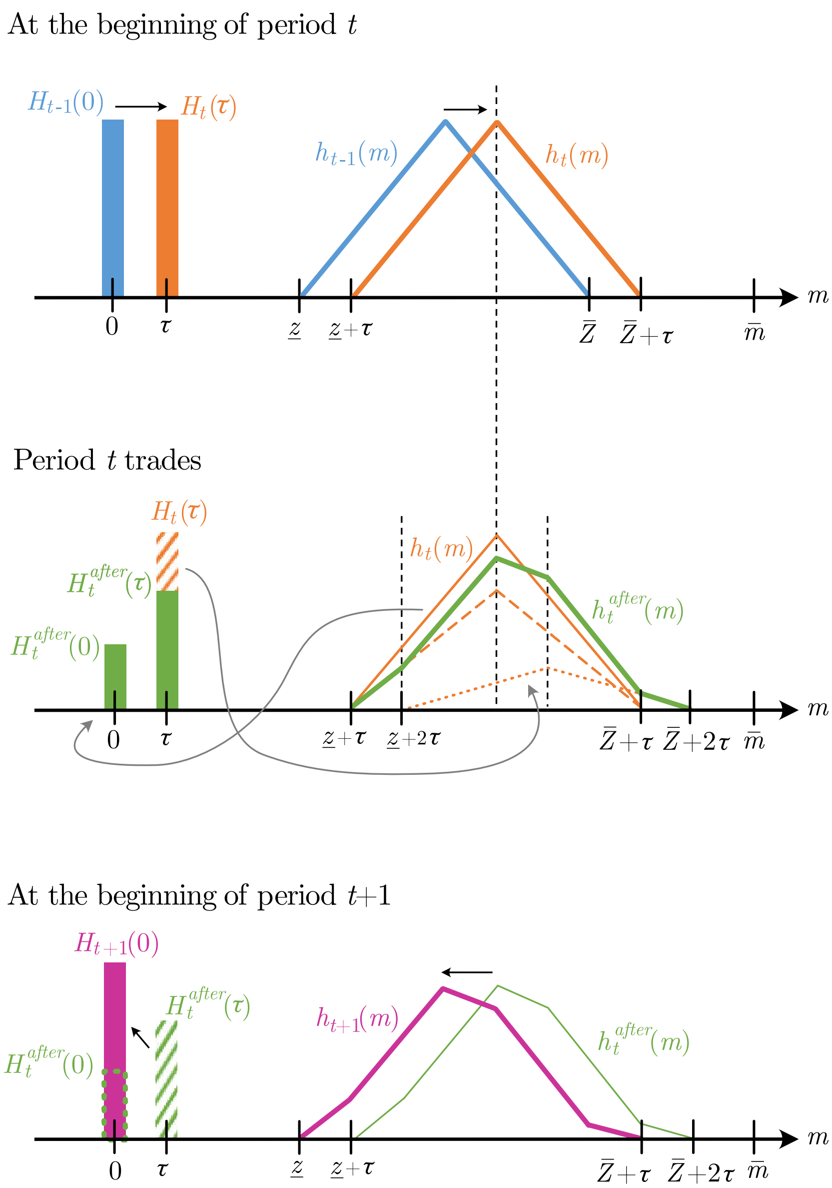
The economy is in a stationary pay-all equilibrium in period $t-1$. At the beginning of period t, the government unexpectedly announces the following policy.

- It injects the same τ unit of money to everybody at the beginning of period t.
- It is going to collect τ unit of money in period $t+1$, if an agent has $m \geq \tau$. If an agent has $m < \tau$, it is going to collect m.

For simplicity, assume that density function $h_t(m)$ associated with H_t exists for $m \in [\underline{z}, \overline{Z}]$. This policy moves the stationary distribution of money holdings to another stationary distribution in one period. An example of a triangle distribution $h_t(m)$ for $[\underline{z}, \overline{Z}]$ is drawn in Figure 6.

On the left-hand side, at the beginning of period t, zero money holders move to $m = \tau$. In period t, some fail to sell goods and remain at $m = \tau$. Additonary, some buyers pay all the money holding and move to $m = 0$. At the beginning of period $t+1$, τ money holders pay tax and move to 0. This transition recovers the stationary pay-all equilibrium again at period $t+1$ because $H_{t-1}(0) = H_{t+1}(0)$. On the right-hand side, the distribution extends

Figure 6: Money-holding distribution under lump-sum transfer with tax collection



Notes: It is an example of the evolution of the money holding distribution, where $h_{t-1}(m) \in [\underline{z}, \overline{Z}]$ is a triangular distribution.

to the right in period t because some sellers holding τ earn money with additional τ held by buyers. This one-time chance to earn extra money makes sellers work hard and improve social welfare.

Proposition 6. *The lump-sum transfer with tax collection improves social welfare if and* *only if*

$$\beta(1-c)\theta^2 > [\theta(1-c)+c](1-\beta)c. \tag{57}$$

Proof. See the Appendix □

Intuitively, this short-run transfer policy makes being a buyer more attractive and encourage seller's production in period t. Under this policy, the number of sellers, buyers, and the matches in period t remain unchanged. Therefore, the improvement in social welfare depends only on the increase in total production per matching. Under inequality (57), buyers' values relative to sellers' are inefficiently small in the stationary equilibrium.[14] The negotiated amount of production $x(m)$ is also small. This policy shifts the money holding distribution at period $t+1$ toward the right. If a seller trades good in period t, this seller will hold relatively more money at $t+1$. Compared to the original steady state, this change increases the discounted sum of utilities of being a buyer in period $t+1$, which leads to higher production. Note that the quantitative welfare improvement by this policy is indeterminate.

Although Proposition 6 shows the direction of the welfare change, its quantitative impact is indeterminate. The seller's incentive depends on the value of extra revenue τ in period t. Its real value is determined by the indeterminate slope of the value functions. Moreover, the seller also cares how much he/she will earn relative to other agents. This effect is affected by the indeterminate shape of the distribution of money holdings.

Our results are parallel to Wallace (2014)'s conjecture. Namely, the optimal monetary policy requires the lump-sum part of transfer schemes. Although we do not directly show the conjecture, our results support the importance of distributive feature of monetary expansion. That is, on the one hand, proportional transfers are neutral under the pay-all equilibrium, and on the other hand, Proposition 6 suggests that a non-linear change in the distribution of money holdings between period $t-1$ and $t+1$ is effective. Further work is needed to establish more direct connections to the conjecture.

[14] Inequality (57) is rewritten as $\frac{(1-\beta)c}{\beta\theta(1-c)} < \frac{\theta}{\theta(1-c)+c}$. From equation (19), we can derive $\frac{\overline{v}_b}{v(0)} = \frac{\theta c\left(\frac{(1-\beta)c}{\beta\theta(1-c)}\right)}{(1-\theta)\left(\alpha-\frac{(1-\beta)c}{\beta\theta(1-c)}\right)}$. Hence, a small enough $\frac{(1-\beta)c}{\beta\theta(1-c)}$ means that $\frac{\overline{v}_b}{v(0)}$ is also small.

8 Discussion

Here, we further analyze the characteristics of the pay-all equilibrium. The first one is about the robustness of pay-all equilibrium by replacing the bargaining protocol with the proportional solution of Kalai (1977b). Second, we prove the equivalence between the axiomatic Nash bargaining solution and the joint surplus maximization under the possible violation of the convexity of the bargaining set in pay-all equilibrium.

8.1 Proportional Solution

Aruoba et al. (2007) show that Lagos and Wright (2005)'s results depend on the choice of bargaining solution. Specifically, they show that changing the Nash bargaining solution to Kalai (1977b)'s proportional solution affects the results both qualitatively and quantitatively. In the following, we show that, in our model, the same changes result in the almost same result.

As in Section 3, we focus on on-path trades, that is, each buyer holds $m \in [\underline{z}, \overline{Z}]$ and each seller has zero. Following Thomson (1994), the proportional solution under the pay-all equilibrium is x in the following equation.

$$(1-\theta_P)\left[k+x+\beta\big(v(0)-v(m)\big)\right] = \theta_P\left[-d-cx+\beta\big(v(m)-v(0)\big)\right], \tag{58}$$

where $\theta_P \in [0,1]$ is the buyer's bargaining power.[15] In other words, the ratio of the buyer's and seller's surpluses is θ_P to $1-\theta_P$. Compared to Nash bargaining equation (4),the difference is in the coefficients of the buyer's surpluses, which is $(1-\theta)c$ under the Nash bargaining but $1-\theta$ under the proportional solution. The two solutions are qualitatively equivalent. By defining

$$\hat{\theta} = \frac{c\theta_P}{1-\theta_P+c\theta_P},$$

(58) can be rewritten as

$$(1-\hat{\theta})c\left[k+x+\beta\big(v(0)-v(m)\big)\right] = \hat{\theta}\left[-d-cx+\beta\big(v(m)-v(0)\big)\right], \tag{59}$$

[15]For a strategic foundation of the proportional solution, see, for example, Hu and Rocheteau (2020).

which is the same as the first order condition of Nash bargaining (4), except that θ is replaced by $\hat{\theta}$. Therefore, $H(0)$, $\overline{x} = \int_{\underline{z}}^{\overline{Z}} x(m)dH/(1-H(0))$, $v(0)$, and $\overline{v}_b = \int_{\underline{z}}^{\overline{Z}} v(m)dH/(1-H(0))$ can be obtained by exactly the same way as in Section 3.

8.2 Consistency with the axiomatic Nash bargaining solution

In the previous sections, we applied the maximization of Nash product without checking the convexity of the bargaining set. Therefore, the solution is possibly different from the axiomatic Nash bargaining solution. The convexity is assumed in the standard proof of the equivalence between the non-symmetric Nash bargaining solution and the point maximizing the non-symmetric Nash product (see, for example, Kalai (1977a)). Specifically, the convexity is used to show that the maximizing point is a unique one satisfying the axiomatic conditions for the non-symmetric Nash bargaining solution. In our case, the bargaining set might be non-convex around the threat point, because there are jumps in the utility and cost functions at $x = 0$ and a jump in the value function at m^1. However, our non-convexity, even if it exists, is not crucial. To prove that the non-symmetric Nash solution coincides with the unique point maximizing the non-symmetric Nash product, Kalai (1977a) uses the convexity only for proving the uniqueness of the point maximizing the Nash product and for the existence of a hyperplane separating the optimal point and the set of points inferior to it.[16] In our case, we can show the uniqueness and existence of a hyperplane using the linearity of the bargaining frontier.

We consider bargaining between a buyer with $m_b \geq m^1$ and a seller with $m_s \in [0, m^1]$. Under the EBOs (see Figure 3), this case covers all possible trades, both on-path and off-path. We respectively define the buyer's and the seller's surpluses as

$$B(p,x) \equiv k + x + \beta(v(m_b - p) - v(m_b)),$$
$$S(p,x) \equiv -d - cx + \beta(v(m_s + p) - v(m_s)).$$

From the following lemma, we can derive the bargaining frontier.

[16]Note that the compactness of the bargaining set is also assumed by Kalai (1977a). This study uses the compactness only for proving the existence of a point maximizing the Nash product. Although the bargaining set might not be closed because of the jumps in our case, existence can be proved, as shown in the proof of Proposition 3.

Lemma 3. *Let*

$$L \equiv \left\{(z_1, z_2) \mid cz_1 + z_2 = \beta\big(v(m_s + m_b) - v(m_s)\big) + c\beta\big(v(0) - v(m_b)\big) + ck - d\right\}.$$

Under the assumptions in Proposition 3, $\big(B(p, x), S(p, x)\big)$ is below L for all $p \in [0, m_b]$ and $x > 0$, that is

$$cB(p, x) + S(p, x) \leq \beta\big(v(m_s + m_b) - v(m_s)\big) + c\beta\big(v(0) - v(m_b)\big) + ck - d.$$

Proof. See the Appendix. □

Under the pay-all property, $p = m_b$ holds, and

$$cB(m_b, x) + S(m_b, x) = \beta\big(v(m_s + m_b) - v(m_s)\big) + c\beta\big(v(0) - v(m_b)\big) + ck - d$$

holds for all $x > 0$, because the coefficient of x in $cB(m_b, x)$ is c and that in $S(m_b, x)$ is $-c$. From Lemma 3, it immediately follows that the bargaining frontier is

$$L^F \equiv \left\{\big(B(m_b, x), S(m_b, x)\big) \mid x > 0, B(m_b, x) \geq 0, S(m_b, x) \geq 0\right\} \subset L.$$

Moreover, hyperplane L separates the set of feasible $\big(B(p, x), S(p, x)\big)$ and the point maximizing the Nash product, which is in L^F and in the relative interior of L, as shown in the proof of Proposition 3. Finally, since L^F is linear, the Nash product is maximized at a unique $\big(B(m_b, x), S(m_b, x)\big)$. Note that another way to deal with the non-convexity is to introduce a lottery, which makes the bargaining set convex. Since the bargaining frontier is linear, the point maximizing the Nash product is not a lottery.

9 Conclusions

We proposed an analytical model of search and bargaining of divisible money. Owing to fixed production costs, the distribution of money holdings is separated into two regions and the associated equilibrium becomes sufficiently tractable for the proof of existence and analytical characterization. The equilibrium is possibly inefficient due to bargaining power parameter; however, it can be improved by redistributive policies.

One important extension is pending in our analysis of monetary expansions. Although we derive analytical results on one case of temporary distributional policy, there are unsolved cases of standard lump-sum transfers on the constant money growth and the one-time helicopter drop. Those obstacles are on the possibility of infinite support and non-separation of the distribution of money holdings. We conjecture that both can be resolved by the introduction of separated buyer/seller sides in the market.

Preference assumptions may also be relaxed. Our pay-all equilibrium is obtained under linear utility and cost functions. However, pay-all equilibrium arises even if these functions are slightly nonlinear or if the fixed terms are significantly large. Moreover, the equilibrium may still be characterized without the pay-all property. The tractability of our model hinges on the separation of the distribution of money holdings to a countable number of regions. Such a distribution may be attained again by the fixed costs that allow buyers to spend money two or more finite times.

Moreover, there is another potential direction towards quantitative studies combined with the real-world distribution of liquidity asset holdings. For example, using Japan's bank account microdata, Kubota et al. (2021) and Kaneda et al. (2021) document that about 20-30% of households live hand-to-mouth in terms of liquid assets. Interestingly, their monthly balances follow our model's alternate transition of money holdings. Quantitative studies possibly solve more general classes of equilibria and provide more realistic implications for policy analyses.

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Appendix

A.1 Proof of Proposition 2

Proof. Over agents $m_t \in [\underline{z}, \overline{Z}]$, fraction $1 - \alpha H(0)$ keeps the same m_t. The other $\alpha H(0)$ fraction is replaced by agents moved from $m_t \in [\zeta \underline{z}, \zeta \overline{Z}]$. Their money holdings follow $X_{s,t}$ and additionally obtain $(1-\zeta)X_{b,t}$ by selling goods. Therefore, $X_{b,t+1}$ is expressed as

$$X_{b,t+1} = \begin{cases} X_{b,t} & \text{with prob. } 1 - \alpha H(0), \\ X_{s,t} + (1-\zeta)X_{b,t} & \text{with prob. } \alpha H(0). \end{cases}$$

Since the matchings are random, $Cov(X_{s,t}, X_{b,t}) = 0$ holds, therefore the variance of $X_{b,t+1}$ is calculated as follows.

$$\begin{aligned} Var[X_{b,t+1}] &= \mathbb{E}\Big[\big(X_{b,t+1} - \tilde{m}_b\big)^2\Big] \\ &= (1 - \alpha H(0))\mathbb{E}\left[(X_{b,t} - \tilde{m}_b)^2\right] + \alpha H(0)\mathbb{E}\left[\Big(X_{s,t} + (1-\zeta)X_{b,t} - \tilde{m}_b\Big)^2\right] \\ &= (1 - \alpha H(0))\mathbb{E}\left[(X_{b,t} - \tilde{m}_b)^2\right] + \alpha H(0)\mathbb{E}\left[\Big((X_{s,t} - \tilde{m}_s) + (1-\zeta)(X_{b,t} - \tilde{m}_b)\Big)^2\right] \\ &= (1 - \alpha H(0))Var(X_{b,t}) + \alpha H(0)Var(X_{s,t}) \\ &\qquad + \alpha H(0)(1-\zeta)^2 Var(X_{b,t}) + \alpha H(0)(1-\zeta)Cov(X_{s,t}, X_{b,t}) \\ &= \big[1 - \alpha H(0)\zeta(2-\zeta)\big]Var(X_{b,t}) + \alpha H(0)Var(X_{s,t}). \end{aligned} \tag{A.1}$$

Next, consider the agents with $m_t \in [\zeta \underline{z}, \zeta \overline{Z}]$. Among them, fraction $1 - \alpha\big(1 - H(0)\big)$ keeps the same m_t, and $\alpha\big(1 - H(0)\big)$ fraction is replaced by agents moved from $m_t \in [\underline{z}, \overline{Z}]$ and they keep ζm_t. Then

$$X_{s,t+1} = \begin{cases} X_{s,t} & \text{with prob. } 1 - \alpha\big(1 - H(0)\big), \\ \zeta X_{b,t} & \text{with prob. } \alpha\big(1 - H(0)\big). \end{cases}$$

Similarly, we can show that

$$Var[X_{s,t+1}] = \big[1-\alpha\big(1-H(0)\big)\big]\,Var(X_{s,t}) + \big[\alpha\big(1-H(0)\big)\zeta^2\big]\,Var(X_{b,t}) \tag{A.2}$$

From equations (A.1) and (A.2), the transition of the two variances can be expressed by a system of linear difference equations:

$$\begin{pmatrix} Var[X_{b,t+1}] \\ Var[X_{s,t+1}] \end{pmatrix} = \Pi \begin{pmatrix} Var[X_{b,t}] \\ Var[X_{s,t}] \end{pmatrix},$$

where

$$\Pi = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} = \begin{pmatrix} 1-\zeta(2-\zeta)\alpha H(0) & \alpha H(0) \\ \alpha\big(1-H(0)\big)\zeta^2 & 1-\alpha\big(1-H(0)\big) \end{pmatrix}.$$

Let the eigenvalues of Π be λ_1 and λ_2 where $\lambda_1 \geq \lambda_2$. The convergence can be proven by $|\lambda_1| < 1$ and $|\lambda_2| < 1$ given a sufficiently small $\zeta > 0$. The eigenvalues are the solutions of

$$(\pi_{11}-\lambda)(\pi_{22}-\lambda) - \pi_{12}\pi_{21} = 0. \tag{A.3}$$

Suppose that $\zeta = 0$. Then $\pi_{21} = 0$, $\lambda_1 = \pi_{11} = 1$ and $\lambda_2 = \pi_{22} = [1-\alpha(1-H(0))] < 1$ hold. It is the original case that $\tilde{H}_{s,t}$ is degenerate at $m = 0$ and $\tilde{H}_{b,t}$ is non-degenerate and stationary.

Next, consider a sufficiently small $\zeta > 0$. We show $\lambda_1 < 1$ by proving that the derivative $\frac{d\lambda}{d\zeta}$ is strictly decreasing around $\zeta = 0$ and $\lambda = 1$. By the implicit function theorem applied to (A.3)

$$\left.\frac{d\lambda}{d\zeta}\right|_{\zeta=0,\lambda=1} = \frac{-\frac{d\pi_{11}}{d\zeta}(\pi_{22}-1) + \pi_{12}\frac{d\pi_{21}}{d\zeta}}{2-(\pi_{11}+\pi_{22})},$$

where $\frac{d\pi_{11}}{d\zeta} = 0$, $\pi_{22} - 1 < 0$, $\pi_{12} > 0$, $\frac{d\pi_{21}}{d\zeta} > 0$, and $2 - (\pi_{11} + \pi_{22}) > 0$. Therefore, $\left.\frac{d\lambda}{d\zeta}\right|_{\zeta=0,\lambda=1} < 0$. Since Π and its characteristic equation are C^1 functions of ζ, then, there exists a $\bar{\zeta} > 0$ such that, for all $\zeta \in (0, \bar{\zeta})$, $Var[X_{s,t}] \to 0$ and $Var[X_{b,t}] \to 0$ as $t \to \infty$. □

A.2 Proof of Proposition 3

The sketch of the proof is as follows.

- *Step 1. Properties of the pay-all equilibrium.* We derive two lemmas to be used in the following discussion. The first is the optimality condition of the Nash bargaining where buyers and sellers hold arbitrary amounts of money (Lemma 4). The second one is about the shape of $v(m)$, that is, the ratio between the two slopes below/above m^1 (Lemma 5).

- *Step 2. Endogenous variables and bargaining outcomes.* We derive the conditions for endogenous variables that satisfy the existence of pay-all equilibria. These are consistent with both on-path and off-path bargaining outcomes (EBOs in Section 4).

- *Step 3. Sufficient parameter conditions.* We show that the parameter conditions in the premises of Proposition 3 are sufficient to support the existence conditions of endogenous variables in *Step 2*.

- *Step 4. Verification.* Finally, we confirm that the candidates for v and H are consistent with the equilibrium, that is, H is stationary and v satisfies Bellman equations (28) and (29).

Step 1. Properties of pay-all equilibrium.

This step first derives some properties of the Nash bargaining solution in the meeting between a buyer holding $m_b \in [0, \overline{m}]$ and a seller holding $m_s \in [0, \overline{m}]$. The optimality condition is derived given a positive amount of trade. Note that, in this step, we do not assume the linearity of v.

Lemma 4. *Consider a bargaining problem between a buyer holding m_b and a seller holding m_s. For a given $p > 0$, suppose the optimal x^* is positive. Then,*

$$cx^* = (1-\theta)c\beta[v(m_b) - v(m_b - p)] + \theta\beta[v(m_s + p) - v(m_s)] - [(1-\theta)ck + \theta d] \quad \text{(A.4)}$$

holds. The seller's surplus is

$$(1-\theta)\big\{\beta[v(m_s+p)-v(m_s)]-c\beta[v(m_b)-v(m_b-p)]+(ck-d)]\big\}, \tag{A.5}$$

and the buyer's surplus is

$$(\theta/c)\big\{\beta[v(m_s+p)-v(m_s)]-c\beta[v(m_b)-v(m_b-p)]+(ck-d)\big\}. \tag{A.6}$$

Proof. Since $x^* > 0$, it is determined by the first-order condition for the Nash bargaining problem with respect to x, that is,

$$-(1-\theta)c[k+x^*+\beta(v(m_b-p)-v(m_b))]+\theta\left[-d-cx^*+\beta\big(v(m_s+p)-v(m_s)\big)\right]=0,$$

which yields (A.4), (A.5), and (A.6). □

Next, we derive a property regarding the shape of the value function. It was guessed as

$$v(m)=\begin{cases}\frac{ck-d}{\beta(1-c)}+am & \text{if } 0\le m<m^1,\\ \frac{ck-d}{\beta(1-c)}+F+bm & \text{if } m^1\le m,\end{cases} \tag{31}$$

where $m^1 < \overline{m}$, $b > a > 0$, and $F > 0$, are endogenous variables. Recall that the candidate for an equilibrium distribution of money holdings H is the one with the support of $\{0\}\cup[\underline{z},\overline{Z}]$ satisfying (1), (2), and (30).

Here, we derive the relationship between slopes a and b. Slope a is the marginal life-time utility of money of an agent being a seller. To use money, this agent needs to first sell the good and then become a buyer. This takes at least one period; hence, the marginal value is discounted by β. By contrast, b is for a potential buyer holding $m \geq m^1$. Because this agent may immediately use money, $b > a$ holds.

Lemma 5. *Suppose that the EBOs in Section 4 are satisfied. Then, the coefficients a and b satisfy*

$$a=\chi b, \tag{A.7}$$

where $\chi \equiv \frac{\alpha\beta(1-\theta)[1-H(0)]}{\{\alpha\beta(1-\theta)[1-H(0)]+1-\beta\}} < 1.$

Proof. Consider an agent holding $m < m^1$. Because this agent sells goods when they become a seller and the partner has $m_b \in [\underline{z}, \overline{Z}]$, by substituting $p(m_b, m_s, H)$ and $x(m_b, m_s, H)$ into the Bellman equation, the value function satisfies

$$v(m) = \alpha \int_{m^1}^{\overline{m}} \Big\{ -d - cx(m_b, m, H) + \beta[v(m + p(m_b, m, H)) - v(m)] \Big\} dH(m_b) + \beta v(m).$$

Substituting (A.4) into the above yields

$$\begin{aligned}&(1-\beta)v(m)\\ &= \alpha(1-\theta) \int_{m^1}^{\overline{m}} \Big\{ \beta[v(m + p(m_b, m, H)) - v(m)] - c\beta[v(m_b) - v(0)] + (ck - d)] \Big\} dH(m_b).\end{aligned} \tag{A.8}$$

By the pay-all property, $\partial p(m_b, m, H)/\partial m = 0$, $\partial v(m)/\partial m = a$, and $\partial v(m+p(m_b, m, H))/\partial m = b$ hold. By taking the first-order derivative with respect to m of both sides of (A.8),

$$(1-\beta)a = \alpha\beta(1-\theta)[1-H(0)](b-a)$$

is obtained and (A.7) holds. □

Step 2. Endogenous variables and bargaining outcomes.

In this step, we derive the conditions for the endogenous variables that satisfy EBOs. The next lemma derives sufficient conditions that bargaining never reaches agreement if either $m_b < m^1$ or $m_s \geq m^1$. These cover all no-agreement cases in the EBOs.

Lemma 6. *Suppose that*

$$\frac{\overline{m}}{2} < m^1 < \overline{m}, \text{ and} \tag{A.9}$$

$$bm^1 < d. \tag{A.10}$$

Then, the bargaining can reach an agreement with $x(m_b, m_s, H) > 0$ *only if*

$$m_s < m^1 \leq m_s + p(m_b, m_s, H). \tag{A.11}$$

It also implies that any buyer with $m_b < m^1$ *does not trade in pay-all equilibria.*

Proof. Suppose (A.11) does not hold and the bargaining reaches an agreement with $x(m_b, m_s, H) > 0$. Then, in the seller's surplus, $v(m_s + p(m_b, m_s, H)) - v(m_s)$ does not contain F.

We first consider the case in which $m_s < m^1$. Because $m^1 \leq m_s + p(m_b, m_s, H)$ does not hold and the increase in the seller's discounted utility depends only on the linear term, the maximum increase in $v(m_s + p) - v(m_s)$ is obtained when acquiring the maximum amount of money, that is, am^1. Therefore, the maximum amount of the surplus does not exceed $-d + am^1 < -d + bm^1$ and is negative because of (A.10). Therefore, the seller's surplus is negative and bargaining does not reach an agreement with $x > 0$. However, this is a contradiction.

Next, we consider case $m_s \geq m^1$. The increase in the seller's discounted utility depends only on the linear term and its maximum increase is obtained when acquiring the maximum amount of money, that is, $b(\overline{m} - m_s) < b(2m^1 - m^1) = bm^1$. Then, the maximum surplus does not exceed $-d + bm^1$ and is negative because of (A.10).

Finally, we consider a buyer with $m_b < m^1$. The buyer's payment satisfies $p \leq m_b < m^1$. Because each seller holds $m_s = 0$ in equilibrium, $m_s + p = 0 + p < m^1$, which violates (A.11). □

Next, we derive the conditions for all agreement cases in the EBOs. Under the conditions of the following lemma, $x(m_b, m_s, H) > 0$ and $p(m_b, m_s, H) = m_b$ hold on the equilibrium path. That is, the buyer is not afraid of the discontinuous decline of $v(m)$ at m^1 by spending all the money holding.

Lemma 7. *Suppose*

$$[(1-\theta)c + \theta]\beta F > (1-\theta)ck + \theta d, \text{ and} \tag{A.12}$$

$$F < \left(\frac{1-c}{c}\right) bm^1. \tag{A.13}$$

Then, $x(m_b, m_s, H) > 0$ and $p(m_b, m_s, H) = m_b$ hold in the bargaining between a seller with $m_s \in [0, m^1)$ and a buyer with $m_b \in [m^1, \overline{m}]$.

Proof. We first show that, given pay-all case $p = m_b$, the surplus at the optimal production, denoted by $x^*_{p=m_b}$, is positive. This means that both agents agree with the trade. Later, we show the optimality of $p = m_b$. From Lemma 4,

$$cx^*_{p=m_b} = [(1-\theta)c + \theta]\beta(F + bm_b) + \theta\beta(b-a)m_s - [(1-\theta)ck + \theta d]$$

holds. Because (A.12) holds and $b > a$ is a property of the candidate value function, $x^*_{p=m_b} > 0$ holds. From (A.5) and (A.6), both seller and buyer surpluses are positive because

$$\begin{aligned} &\beta[v(m_s + m_b) - v(m_s)] - c\beta[v(m_b) - v(0)] + (ck - d) \\ &= (1-c)\beta(F + bm_b) + \beta(b-a)m_s + (ck - d) > 0. \end{aligned} \tag{A.14}$$

Whether $p = m_b$ is optimal or not, the optimal solution provides positive surpluses that are equal to or higher than (A.14). Therefore, $x(m_b, m_s, H) > 0$ holds.

Then, from Lemma 6, we can focus on case $m_s + p \geq m^1$. For a buyer, we consider the regions of p satisfying (a) $m_b - p < m^1$ and (b) $m_b - p \geq m^1$. Below, we show that the Nash product is maximized at $p = m_b$ in region (a). In addition, we prove that the Nash product at $p = m_b$ in region (a) is larger than the Nash product at any p in region (b), that is, $p(m_b, m_s, H) = m_b$.

In region (a), we show that the derivative of the Nash product with respect to p is always positive, which implies the optimality of $p = m_b$. We define the buyer's surplus as $B \equiv k + x + \beta\big(v(m_b - p) - v(m_b)\big)$ and the seller's surplus as $S \equiv -d - cx + \beta\big(v(m_s + p) - v(m_s)\big)$. By $m_s + p \geq m^1$, $m_b - p < m^1$, and (31), $\partial v(m_s + p)/\partial p = b$ and $\partial v(m_b - p)/\partial p = -a$. Then, the derivative of the Nash product $B^\theta S^{1-\theta}$ with respect to p is

$$\frac{\partial}{\partial p}\big(B^\theta S^{1-\theta}\big) = \beta B^{\theta-1} S^{-\theta}[b(1-\theta)B - a\theta S].$$

The first-order condition of Nash bargaining problem with respect to x leads to $\theta S = (1-\theta)cB$. Thus,

$$\frac{\partial}{\partial p}\big(B^\theta S^{1-\theta}\big) = \beta B^\theta S^{-\theta}(b - ca)(1-\theta) > 0. \tag{A.15}$$

This implies that Nash product reaches its maximum at the upper bound, $p = m_b$.

Next, we consider region (b). Suppose, on the contrary, that the optimal $p(m_b, m_s, H)$ is in region (b), that is, $m_b - p(m_b, m_s, H) \geq m^1$. Because $x(m_b, m_s, H) > 0$, we can apply Lemma 4 for the optimal allocation. From (A.5), seller's surplus S satisfies

$$S = (1-\theta)\left\{\beta[F + (b-a)m_s] + b\beta(1-c)p(m_b, m_s, H) + (ck - d)\right\}.$$

By $p(m_b, m_s, H) \leq m_b - m^1$,

$$S \leq (1-\theta)\left\{\beta[F + (b-a)m_s] + b\beta(1-c)\left(m_b - m^1\right) + (ck - d)\right\}. \tag{A.16}$$

The total surplus is the term after $(1-\theta)$ in (A.16). Under (A.13), it is dominated by (A.14), which is the maximum surplus in region (a). Therefore, the optimal $p(m_b, m_s, H)$ is not in region (b) but in region (a), and $p(m_b, m_s, H) = m_b$ holds. $\square$

Step 3. Sufficient parameter conditions.

In Lemmas 6 and 7, we have shown the relationships of the endogenous variables consistent with the EBOs. In this step, we show that the parameter restrictions of Proposition 3 derive those relationships.

Lemma 8. *Under the assumptions in Proposition 3, there exists a continuum of* (m^1, H, a, b, F) *satisfying the premises of Lemmas 6 and 7, the candidate distribution* H *in* (1) *and* (2)*, and value function* v *in* (31)*.*

Proof. The premises and the properties are summarized as

- (A.9): $\frac{\overline{m}}{2} < m^1 < \overline{m}$,
- (A.10): $bm^1 < d$,
- (A.12): $[(1-\theta)c + \theta]\beta F - (1-\theta)ck - \theta d > 0$,
- (A.13): $F < \left(\frac{1-c}{c}\right) bm^1$,
- (1) and (2): $[\underline{z}, \overline{Z}] \subset [m^1, \overline{m}]$, $\int_{\underline{z}}^{\overline{Z}} dH(m) = 1 - H(0)$ and $\int_{\underline{z}}^{\overline{Z}} m dH(m) = M$.

The main inequalities we need to derive are (A.9), (A.10), (A.12), and (A.13). First, we rewrite these conditions as inequalities with parameters m^1 and b. To do so, we assume the pay-all property and $x(0, m, H) > 0$ for $m \in [m^1, \overline{m})$. Note that the pay-all property and $x(0, m, H) > 0$ will be verified because we will find (m^1, H, a, b, F) satisfying all the premises in Lemma 7. From (13),

$$\overline{v}_b = \frac{1}{1 - H(0)} \int_{\underline{z}}^{\overline{Z}} \big(v(0) + F + bm\big) dH(m) = v(0) + F + \frac{bM}{1 - H(0)}.$$

Therefore,

$$F = \overline{v}_b - v(0) - \frac{bM}{1 - H(0)}. \tag{A.17}$$

Note that b and F are indeterminate, but each pair satisfies (A.17). Therefore, F is determined for a given b. By using (A.17), we can eliminate F from conditions (A.10), (A.12), and (A.13).

$$\begin{aligned} b &< \frac{d}{m^1}, && \text{(A.18)}\\ b &< \left(\frac{1 - H(0)}{M}\right)\left[\overline{v}_b - v(0) - \frac{(1-\theta)ck + \theta d}{\beta[(1-\theta)c + \theta]}\right], && \text{(A.19)}\\ b &> \frac{\overline{v}_b - v(0)}{\frac{(1-c)m^1}{c} + \frac{M}{1-H(0)}}. && \text{(A.20)} \end{aligned}$$

The existence of b requires the right-hand side of (A.20) to be smaller than those of (A.18) and (A.19), and written as

$$\begin{aligned} \frac{\overline{v}_b - v(0)}{\frac{(1-c)m^1}{c} + \frac{M}{1-H(0)}} &< \frac{d}{m^1} && \text{(A.21)}\\ \frac{\overline{v}_b - v(0)}{\frac{(1-c)m^1}{c} + \frac{M}{1-H(0)}} &< \left(\frac{1 - H(0)}{M}\right)\left[\overline{v}_b - v(0) - A\right], && \text{(A.22)} \end{aligned}$$

where $A = \frac{(1-\theta)ck+\theta d}{\beta[(1-\theta)c+\theta]}$. Note that $A < F$ holds by (A.12).

We now consider the conditions that m_1 satisfies the above inequalities. The right-hand side of (A.22) is positive because, from (A.17),

$$\overline{v}_b - v(0) - A > \overline{v}_b - v(0) - F = \frac{bM}{1 - H(0)} > 0.$$

Then, we rewrite (A.22) as

$$m^1 > \left(\frac{M}{1-H(0)}\right)\left(\frac{Ac}{(1-c)(\overline{v}_b - v(0) - A)}\right), \tag{A.23}$$

Inequality (A.21) is equivalent to

$$\overline{v}_b - v(0) - \frac{d(1-c)}{c} < \left(\frac{M}{1-H(0)}\right)\left(\frac{d}{m^1}\right). \tag{A.24}$$

As for (A.9), we rewrite it and add condition $m^1 < \frac{M}{1-H(0)}$, which is for the existence of H satisfying (1) and (2), as follows.

$$\frac{\overline{m}}{2} < m^1 < \frac{M}{1-H(0)} < \overline{m}. \tag{A.25}$$

Under (A.25), we can find $\underline{z} \leq \overline{Z}$ and H satisfying $\int_{\underline{z}}^{\overline{Z}} m dH(m) = M$. To summarize, if there exists m^1 that satisfies (A.23), (A.24), and (A.25), there also exists b satisfying (A.10), (A.12), and (A.13), and equilibrium exists.

There are two possible cases. First, suppose

$$\overline{v}_b - v(0) \leq \frac{d(1-c)}{c}. \tag{A.26}$$

Then (A.24) is immediately satisfied. Since $\overline{m}/2 < M/(1-H(0))$ is assumed in Proposition 3, then, if

$$\left(\frac{M}{1-H(0)}\right)\left(\frac{Ac}{(1-c)(\overline{v}_b - v(0) - A)}\right) < \frac{M}{1-H(0)} \tag{A.27}$$

is satisfied, taking m^1 close enough to $M/(1-H(0))$ all inequalities hold. We can rewrite inequality (A.27) as $A/(1-c) < \overline{v}_b - v(0)$. Together with (A.26), this condition is equivalent to (36) in Proposition 3.

Next, suppose $\overline{v}_b - v(0) > \frac{d(1-c)}{c}$. Inequalities (A.23), (A.24), and (A.25) allow the existence of m^1 if

$$\max\left[\frac{Ac}{(1-c)(\overline{v}_b - v(0) - A)}, \left(\frac{M}{1-H(0)}\right)^{-1}\frac{\overline{m}}{2}\right] < \min\left[\frac{d}{\overline{v}_b - v(0) - \frac{d(1-c)}{c}}, 1\right].$$

This is (37) in Proposition 3.

□

Step 4: Verification

Below, we show that candidate distribution H and value function v in (31) satisfy the stationarity of the distribution of money holdings (Lemma 9) and Bellman equations (28) and (29) (Lemmas 10 and 11). In the following three lemmas, we assume the conditions in Proposition 3; thus, all lemmas in the previous steps can be used.

Lemma 9. *The distribution H defined by* (1) *and* (2) *is stationary.*

Proof. The support for the distribution of money holdings is $\{0\} \cup [\underline{z}, \overline{Z}]$. From Lemma 7, in each trade, a buyer with $m = 0$ and a seller holding $m \in [\underline{z}, \overline{Z}]$ trade. Moreover, the pay-all property holds and after the trade, the seller will hold m and the buyer will have no money in the next period. Therefore, the distribution of money holdings remains the same. □

Lemma 10. *The candidate value function v defined in* (31) *for $m \in [m^1, \overline{m}]$ is the solution to* (28).

Proof. We substitute (31) into the right-hand side of equation (28) and then check that it is indeed the left-hand side of (31). Equation (28) can be rewritten as

$$v(m) = \alpha H(0)\{k + x(m) + \beta[v(0) - v(m)]\} + \beta v(m). \tag{A.28}$$

By the pay-all property, this buyer pays m in each matching. From (31) and (A.6), this buyer's surplus is

$$\begin{aligned} & k + x(m) + \beta[v(0) - v(m)] \\ &= \frac{\theta}{c}\big\{\beta[v(m) - v(0)] - c\beta[v(m) - v(0)] + ck - d\big\} \\ &= \frac{\theta}{c}\left[(1-c)\beta(F + bm) + (ck - d)\right]. \end{aligned}$$

Therefore, by using (31), the right-hand side of equation (A.28) is written as

$$\alpha H(0)\frac{\theta}{c}\left[(1-c)\beta(F+bm)+(ck-d)\right]+\beta\left(bm+F+\frac{ck-d}{\beta(1-c)}\right)$$

$$=\alpha\left(\frac{(1-\beta)c}{\alpha\beta\theta(1-c)}\right)\frac{\theta}{c}\left[(1-c)\beta(F+bm)+(ck-d)\right]+\beta\left(bm+F+\frac{ck-d}{\beta(1-c)}\right) \qquad \text{(from Lemma 1)}$$

$$=(1-\beta)\left(F+bm+\frac{ck-d}{\beta(1-c)}\right)+\beta\left(bm+F+\frac{ck-d}{\beta(1-c)}\right)$$

$$=bm+F+\frac{ck-d}{\beta(1-c)}.$$

The last line is $v(m)$ in (31) for $m\in[m^1,\overline{m}]$, that is, (31) for $m\in[m^1,\overline{m}]$ is a solution to (28). □

Lemma 11. *The candidate value function v defined in* (31) *for $m\in[0,m^1)$ is a solution to* (29).

Proof. We first consider the seller's surplus when we use (31). Let $m_b\in[\underline{z},\overline{Z}]$ and consider the bargaining between m_b and $m_s=m$. From (A.5),

$$-d-cx(m_b,m,H)+\beta[v(m)-v(0)]$$

$$=(1-\theta)[(1-c)\beta(F+bm_b)+\beta(b-a)m]+ck-d] \qquad \text{(by (31))}$$

$$=(1-\theta)\left[(1-c)\beta\left(bm_b+F+\frac{ck-d}{\beta(1-c)}\right)+\beta(b-a)m-(1-c)\beta\frac{ck-d}{\beta(1-c)}+ck-d\right]$$

$$=(1-\theta)\left[(1-c)\beta\left(bm_b+F+\frac{ck-d}{\beta(1-c)}\right)+\beta(b-a)m\right]. \qquad \text{(A.29)}$$

Using (31) and (A.29), the right-hand side of (29),

$$\alpha\int_{\underline{z}}^{\overline{Z}}\{-d-cx(m_b,m,H)+\beta[v(m+m_b)-v(m)]\}dH(m_b)+\beta v(m),$$

is equal to

$$=\alpha\int_{\underline{z}}^{\overline{Z}}(1-\theta)\left[(1-c)\beta\left(bm_b+F+\frac{ck-d}{\beta(1-c)}\right)+\beta(b-a)m\right]dH(m_b)+\beta\left(am+\frac{ck-d}{\beta(1-c)}\right)$$

$$= \alpha(1-\theta)(1-c)\beta\big(1-H(0)\big)\overline{v}_b + \alpha(1-H(0))(1-\theta)\beta(b-a)m + \beta\left(am + \frac{ck-d}{\beta(1-c)}\right)$$

$$= \alpha\left(\frac{1-c}{c}\right)H(0)\theta\beta\frac{ck-d}{\beta(1-c)} + (1-\beta)am + \beta\left(am + \frac{ck-d}{\beta(1-c)}\right) \qquad \text{(by (19) and (A.7))}$$

$$= am + \frac{ck-d}{\beta(1-c)}.$$

The last line is $v(m)$ in (31) for $m \in [0, m^1]$, that is, (31) for $m \in [0, m^1)$ is a solution to (29).

□

A.3 Derivation of $g(m)$ in equation (42)

Assume the linearity of $v(m)$, which can be written as $v(m) = v(0) + F + bm$ as in (31). Below, we show that $g(m)$ is also linear for $m \in [\underline{z}, \overline{Z}]$.

The first-order condition of Nash bargaining problem (43) is

$$(1-\theta)(c+\beta tb)\left[k+x-\beta(F+bm)\right] = \theta\left[-d-cx+\beta\Big(F+b\big(m-tx-g(m)\big)\Big)\right],$$

which is rewritten as

$$\begin{aligned}(c+\beta tb)x = &\big[(1-\theta)(c+\beta tb)+\theta\big]\beta bm - \beta\theta bg(m)\\ &+(1-\theta)(c+\beta tb)(\beta F-k)+\theta(\beta F-d).\end{aligned}$$

Solving x as a function of m and $g(m)$ yields

$$x\big(m, g(m)\big) = Am - Bg(m) + C,$$

where $A = \big[(1-\theta)(c+\beta tb)+\theta\big]\beta b/(c+\beta tb)$, $B = \beta\theta b/(c+\beta tb)$, and $C = \big[(1-\theta)(c+\beta tb)(\beta F-k)+\theta(\beta F-d)\big]/(c+\beta tb)$. Given t, the government solves $g(m) = -tx\big(m, g(m)\big)$ for $g(m)$. That is, from

$$g(m) = -t(Am - Bg(m) + C),$$

the government obtains

$$g(m) = -\frac{tAm + tC}{1 - tB},$$

which is a linear function of m.

A.4 Proof of Proposition 6

Proof. Given the pay-all equilibrium, the transition of the distribution can be described as follows. In period $t-1$ and before, the money holding distribution is stationary with

- $H_{t-1}(0) = H(0) = \frac{(1-\beta)c}{\alpha\beta\theta(1-c)}$,
- $h_{t-1}(m) = h(m) \geq 0$ for all $m \in [\underline{z}, \overline{Z}]$.

At the beginning of period t, τ is injected to everybody. The distribution shifts to the right by τ.

- $H_t(\tau) = H_{t-1}(0)$,
- $h_t(m) = h_{t-1}(m-\tau)$ for all $m \in [\underline{z}+\tau, \overline{Z}+\tau]$.

In period t, suppose that the pay-all property still holds. Then, each buyer with $m \in [\underline{z}+\tau, \overline{Z}+\tau]$ pays all the money holding. Each seller finds a buyer with probability $\alpha\big(1-H_t(\tau)\big)$, and each buyer finds a seller with probability $\alpha H_t(\tau)$. Let H_t^{after} be the money holding distribution after the trade at period t, and is derived as follows.

- $m = 0$: buyers who spend all money holding,

$$H_t^{after}(0) = \alpha H_t(\tau)\big(1-H_t(\tau)\big) = \alpha H_{t-1}(0)\big(1-H_{t-1}(0)\big).$$

- $m = \tau$: sellers who do not find buyers,

$$H_t^{after}(\tau) = H_t(\tau)\left[1-\alpha\big(1-H_t(\tau)\big)\right] = H_{t-1}(0)\left[1-\alpha\big(1-H_{t-1}(0)\big)\right].$$

- $m \in [\underline{z}+\tau, \underline{z}+2\tau)$: buyers hold $m \in [\underline{z}+\tau, \underline{z}+2\tau)$ at the beginning of period t and do not find sellers,

$$h_t^{after}(m) = h_t(m)\big(1-\alpha H_t(\tau)\big) = h_{t-1}(m-\tau)\big(1-\alpha H_{t-1}(0)\big).$$

- $m \in [\underline{z}+2\tau, \overline{Z}+\tau]$: two groups are possible. One are the buyers who hold $m \in [\underline{z}+2\tau, \overline{Z}+\tau]$ at the beginning of period t and do not find sellers. The other group are the sellers who meet buyers holding $m \in [\underline{z}+\tau, \overline{Z}]$ at the beginning of period t.

$$\begin{aligned} h_t^{after}(m) &= h_t(m)\big(1-\alpha H_t(\tau)\big) + \alpha H_t(\tau) h_t(m-\tau) \\ &= h_{t-1}(m-\tau)\big(1-\alpha H_{t-1}(0)\big) + \alpha H_{t-1}(0) h_{t-1}(m-2\tau). \end{aligned}$$

- $m \in (\overline{Z}+\tau, \overline{Z}+2\tau]$: sellers who meet buyers holding $m \in (\overline{Z}, \overline{Z}+\tau]$ at the beginning of period t.

$$h_t^{after}(m) = \alpha H_t(\tau) h_t(m-\tau) = \alpha H_{t-1}(0) h_{t-1}(m-2\tau).$$

At the beginning of period $t+1$, all agents except $m=0$ return τ money to the government.

- $m=0$: non-money holders and agents holding τ at the end of period t:

$$\begin{aligned} H_{t+1}(0) &= H_t^{after}(0) + H_t^{after}(\tau) \\ &= \alpha H_{t-1}(0)\big(1-H_{t-1}(0)\big) + H_{t-1}(0)\big[1-\alpha\big(1-H_{t-1}(0)\big)\big] \\ &= H_{t-1}(0) = H(0). \end{aligned}$$

- $m \in [\underline{z}, \overline{Z}+\tau]$: for all agents in this category, $h_{t+1}(m) = h_t^{after}(m-\tau)$ holds.

$$h_{t+1}(m) = \begin{cases} h_{t-1}(m)[1-\alpha H_{t-1}(0)] & \text{if } m \in [\underline{z}, \underline{z}+\tau) \\ h_{t-1}(m)[1-\alpha H_{t-1}(0)] + \alpha H_{t-1}(0) h_{t-1}(m-\tau) & \text{if } m \in [\underline{z}+\tau, \overline{Z}] \\ \alpha H_{t-1}(0) h_{t-1}(m-\tau) & \text{if } m \in (\overline{Z}, \overline{Z}+\tau] \end{cases} \tag{A.30}$$

Compared to $H_{t-1}(m)$, the new money holding distribution holds the same population of non-money holders: $H_{t+1}(0) = H_{t-1}(0)$. However, the distribution of positive money holders $h_{t+1}(m) \geq \underline{z}$ stretches out to the right. Note again that the pay-all property holds, because $H_{t+1}(0) = H_{t-1}(0) = \frac{(1-\beta)c}{\alpha\beta\theta(1-c)}$, and $h_{t+1}(m)$ can have any shape for $m \in [m^1, \overline{m}]$ under Proposition 3. Hence, the new stationary money holding distribution $H_{t+1}(m) = H_{t+2}(m) = \cdots$ can also hold in the pay-all equilibrium if τ is sufficiently small. By Proposition 1, the macro-level variables and social welfare are the same before $t-1$ and after $t+1$. Therefore, at the macro-level, the policy makes one-period deviation from the same steady state. However, at the micro-level, the transition path converges to a different steady state in one period.

Given the one-period transition path of the money holdings, we derive the allocation of goods using value functions and Nash bargaining solutions. The stationarity implies that $v_{t-1}(0) = v_{t+1}(0) = v(0)$. However, $v_{t-1}(m)$ and $v_{t+1}(m)$ can be different for $m \geq \underline{z}$ by the indeterminacy. At the beginning of period t, each seller holds τ amount of money. The Nash bargaining problem at period t between a seller holding τ and a buyer holding m_t is

$$\max_{x_t} \left[k + x_t + \beta\big(v(0) - v_{t+1}(m_t - \tau)\big)\right]^{\theta} \cdot \left[-d - cx_t + \beta\big(v_{t+1}(m_t) - v(0)\big)\right]^{1-\theta}. \quad \text{(A.31)}$$

Here, the pay-all property still holds at period t, that is, the buyer pays an $m_t + \tau$ amount of money. The agents expect that τ unit will be collected by the government at the beginning of $t+1$. If the bargaining fails, the buyer will hold $m_t - \tau$ at the beginning of period $t+1$, and the seller will have no money. If τ is sufficiently small, then pay-all equilibria exist, because the conditions in Proposition 3 are strict inequalities. Moreover, for a sufficiently small τ, agents having τ money cannot be a buyer and purchase goods because of fixed cost d. By rearranging the first-order condition and replacing m_t by $m_{t-1} + \tau$, we get

$$cx_t(m_{t-1}) = \beta\left[\theta v_{t+1}(m_{t-1} + \tau) + (1-\theta)cv_{t+1}(m_{t-1})\right] - \beta[(1-\theta)c + \theta]v(0) - [(1-\theta)ck + \theta d],$$

Because each buyer finds a seller with probability $\alpha H(0)$, the total production at period t is defined as $Y_t \equiv \alpha H(0) \int_{\underline{z}}^{\overline{Z}} x_t(m) dH_{t-1}(m)$. Then,

$$\begin{aligned} Y_t =& \frac{\alpha\beta H(0)}{c} \int_{\underline{z}}^{\overline{Z}} \left[\theta v_{t+1}(m+\tau) + (1-\theta)cv_{t+1}(m)\right] h_{t-1}(m) dm \\ &- \frac{\alpha H(0)[1 - H(0)]}{c} \left\{\beta[(1-\theta)c + \theta]v(0) + (1-\theta)ck + \theta d\right\} \end{aligned} \quad \text{(A.32)}$$

Now, we will show a condition that Y_t is larger than the steady-state total production $Y = \alpha H(0) \int_{\underline{z}}^{\overline{Z}} x(m) dH(m)$. Then, by the linearities of utility and cost functions, $Y_t > Y$ also means an improvement in social welfare. Since the economy reaches to a steady state at period $t+1$, from equation (5)

$$Y = Y_{t+1} = \frac{\alpha\beta H(0)}{c}\int_{\underline{z}}^{\overline{Z}+\tau}[\theta+(1-\theta)c]v_{t+1}(m)h_{t+1}(m)dm \\ - \frac{\alpha H(0)[1-H(0)]}{c}\left\{\beta[(1-\theta)c+\theta]v(0)+[(1-\theta)ck+\theta d]\right\}. \tag{A.33}$$

From equations (A.32) and (A.33), $Y_t > Y$ is equivalent to

$$\theta\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m+\tau)dh_{t-1}(m)dm + (1-\theta)c\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m)dh_{t-1}(m)dm \\ > [\theta+(1-\theta)c]\int_{\underline{z}}^{\overline{Z}+\tau} v_{t+1}(m)h_{t+1}(m)dm \\ = [\theta+(1-\theta)c]\alpha H(0)\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m+\tau)h_{t-1}(m)dm \\ + [\theta+(1-\theta)c]\big(1-\alpha H(0)\big)\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m)h_{t-1}(m)dm, \tag{A.34}$$

where the last equality is derived by equation (A.30). By $v_{t+1}(m+\tau) > v_{t+1}(m)$, equation (A.34) implies that $Y_t > Y$ is equivalent to $\theta > [\theta+(1-\theta)c]\alpha H(0)$, which can be rewritten as condition (57). □

A.5 Proof of Lemma 3

Proof. As shown in Lemma 8, the assumptions in Proposition 3 imply those in Lemma 7. Therefore, the pay-all property holds. Define $T(p,x) \equiv cB(p,x) + S(p,x)$. Then, from Lemma 4,

$$T(m_b,x^*) = cB(m_b,x^*) + S(m_b,x^*) = \beta\big(v(m_s+m_b) - v(m_s)\big) + c\beta\big(v(0) - v(m_b)\big) + ck - d, \tag{A.35}$$

where the Nash product is maximized at x^* for $p = m_b$. Note that the right-hand side of (A.35) is independent of x^* due to linearities of the utility and cost functions.

Suppose, on the contrary, that there exist $\hat{p} \in [0, m_b]$ and $\hat{x} > 0$ satisfying $T(\hat{p},\hat{x}) > T(m_b,x^*)$. Then, $T(\hat{p},\hat{x}^*) > T(m_b,x^*)$, where the Nash product is maximized at $\hat{x}^*$ for $p = \hat{p}$. Lemma 4 also implies that

$$B(\hat{p},\hat{x}^*) = (\theta/c)T(\hat{p},\hat{x}^*) > (\theta/c)T(m_b,x^*) = B(m_b,x^*),$$
$$S(\hat{p},\hat{x}^*) = (1-\theta)T(\hat{p},\hat{x}^*) > (1-\theta)T(m_b,x^*) = S(m_b,x^*).$$

These inequalities contradict the fact that (m_b,x^*) maximizes the Nash product $B(p,x)^{\theta}S(p,x)^{1-\theta}$. Therefore, for all $p \in [0,m_b]$ and $x > 0$,

$$T(p,x) \leq T(m_b,x^*) = \beta\big(v(m_s+m_b) - v(m_s)\big) + c\beta\big(v(0) - v(m_b)\big) + ck - d$$

□

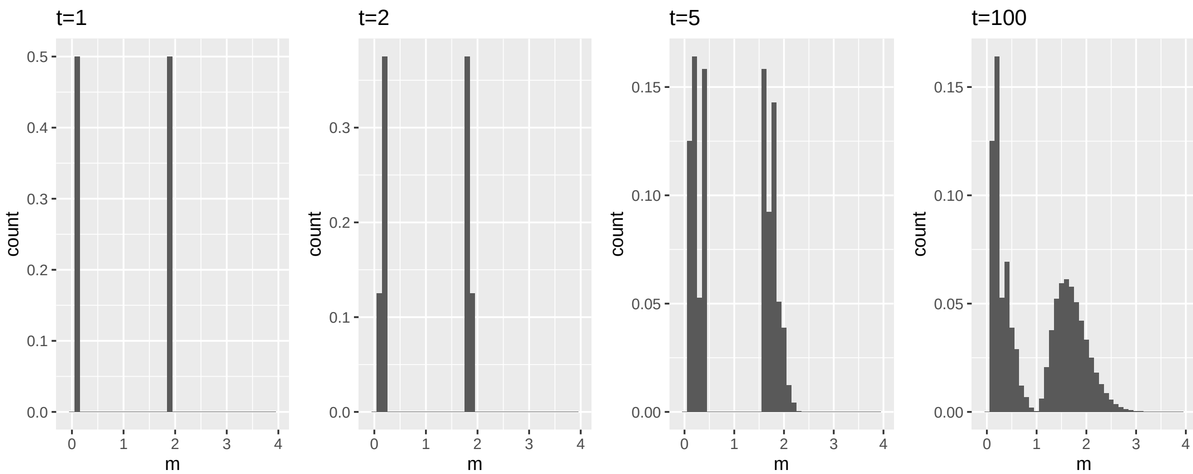
A.6 Lump-sum Transfers

In Section 7, we noted that the straightforward introduction of lump-sum transfers requires some advancements for equilibrium analysis. Below, we consider two major cases.

Under the constant money growth, each agent receives $\tau_t = \mu M_t$ for all t given a fixed money growth rate μ. Figure 7 simulates a transition of the distribution of money holdings starting from the two-point distribution, where the x-axis is the real money holding m_t/M_t. We assume the cut-off point between seller/buyer choice as $m_t^1/M_t = 1$. In this simulation, the support of distributions is separated exactly at this point. Therefore, the cut-off is critically connected to the equilibrium equations and should be determined endogenously. However, this condition is not considered in the original model because the distribution has no measure around the cut-off point. In fact, it is indeterminate in the original model. Hence, the extension to lump-sum transfer needs additional conditions for the cut-off. Note also that the distribution of real balances is bounded because the inflation tax is larger than the lump-sum transfer for large money holders.

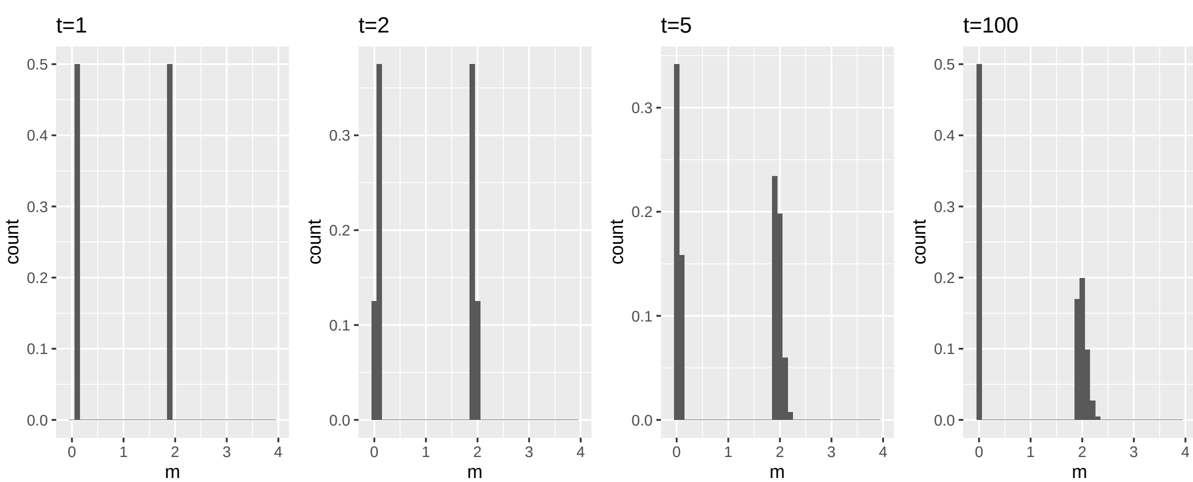
Next, we consider the one-time transfer in Figure 8. Suppose $\mu_t = 0$ for all $t \leq T-1$ and the money supply is constant at M. Then, unexpectedly, μ_T increases to a positive number in period T. Each agent receives $\overline{\tau} = \mu_T M$. After that, $\mu_t = 0$ for all $t \geq T+1$. At time T, the distribution of the money holdings shifts to the right and the maximum amount of money holding increases to $\overline{Z} + \overline{\tau}$. In the next period, all sellers hold $\overline{\tau}$ and some make a revenue $\overline{Z} + \overline{\tau}$. Then, the maximum amount becomes $\overline{Z} + 2\overline{\tau}$. Since some sellers still hold $\overline{\tau}$ in period $T+1$, by the same logic, $\max m_{T+2} = \overline{Z} + 3\overline{\tau}$. Eventually, $\lim_{t\to\infty} m_t = \infty$ and it violates the assumption about the maximum money holding $\overline{m}$. In our conjecture, the distribution will converge to one with a range $\{0\} \cup [\underline{z} + \bar{\tau}, \infty)$. Although the upper-bound $\overline{m}$ matters only off-path, we may need another assumption to eliminate it for the proof of existence.

Figure 7: The distribution under repeated lump-sum transfers with constant money growth



Note: The initial distribution in period 0 is $H_0(0) = H_0(2) = 1/2$. From the left, we plot histograms in period $1, 2, 5$, and 100. The parameters are $\mu = 0.1$ and $\alpha = 0.5$. Moreover, this simulation assumes the seller/buyer cutoff as $m_t^1/M_t = 1$. The x-axis is the real money balance m_t/M_t and each bar width is 0.1.

Figure 8: The distribution responding to one-time lump-sum transfer



Note: The initial distribution in period 0 is $H_0(0) = H_0(2) = 1/2$. From the left, we plot histograms in period $1, 2, 5$, and 100. The parameters are $\mu = 0.1$ and $\alpha = 0.5$. The x-axis is the real money balance m_t/M_t and each bar width is 0.1.