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Zhen Zhao

Graduate School of Economics and Management, Tohoku University

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Zhen Zhao*

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Abstract

We study the problem of allocating a single indivisible good to at most one of n agents when the preferences of agents' are quasilinear, monetary transfers are allowed and strategy-proof mechanism is needed. In this paper, we consider the possibility of constructing feasible allocation mechanisms which satisfy strategy-proofness, anonymity, budget balance and no wastage. In two and three agents cases, we show an impossibility result.

Keywords: Allocation problem; Indivisible good; Feasibility; Strategy-proofness; Anonymity; Budget balance

JEL Classification Numbers: D63; D71; D82.

*Corresponding author.
Faculty of Economics, Tohoku University, Sendai, Miyagi 980-8576, Japan; e-mail: zzjjf113@gmail.com

1 Introduction

Suppose that there is a single indivisible good in the economy. The problem we are facing is to allocate the indivisible good to at most one of n agents. We assume that agents' preferences are quasilinear and monetary transfers are possible. In our model, we also assume that the number of agents is two or three and each agent's valuation of the indivisible good is a positive real number.

Our goal is to examine if there is any desirable allocation mechanism taking full account of fairness and improving welfares of agents. In this paper, we consider the following axioms as the desirable properties. The first axiom is strategy-proofness. Since valuations are regarded as private information, agents have the incentive to manipulate the mechanism. Strategy-proofness requires that truthful revelation of a valuation is always a weakly dominant strategy for each agent. The next one is anonymity. Anonymity requires that agents should be treated fairly and the consumption bundles that agents receive are irrelevant to the indexes of agents. The third one is budget balance. Budget balance requires that the total amount of monetary transfers is always zero. Finally, we consider no wastage which states that leaving the good unallocated forever is not an option.

On the allocation problem, most of the researches focus on the class of Groves (1973) mechanisms.[1] Holmström (1979) shows that the class of Groves mechanisms is the only class of mechanisms which satisfies strategy-proofness and decision-efficiency. Decision-efficiency requires that the good never remains unallocated and must be allocated to the agent who has the highest valuation. Among all the Groves mechanisms, Vickery (1961) mechanism is one of the most famous mechanisms which does not satisfy budget balance. Indeed, budget balance is incompatible with strategy-proofness and decision-efficiency, although it is a good property which is intended for improving welfares of the agents. Green and Laffont (1977) prove that Groves mechanisms do not satisfy budget balance. Kato et al. (2015) drop decision- efficiency and prove that there is no mechanism which is strategy-proof, symmetric and budget balanced. In their model, it is assumed that the indivisible good must be allocated to one agent.

Sprumont (2013) studies the model which allows a single indivisible good to be unallocated and constructs a class of non-Groves mechanisms which are Pareto-optimal in the class of feasible, strategy-proof, anonymous and envy-free mechanisms.[2] One feature of the non-Groves mechanisms is that they are all feasible mechanisms which leave an indivisible good unallocated at some profiles of valuations. However, these non-Groves mechanisms still do not satisfy budget balance. Hence it is a question needed to be answered that if there is a feasible mechanism which is strategy-proof, anonymous, budget balanced and no wasteful.

In this paper, we consider the possibility of constructing feasible allocation mechanisms which satisfy strategy-proofness, anonymity, budget balance and no wastage. Furthermore, we prove that there is no such mechanism. Since the impossibility result proved by Kato et al. (2015) plays an important role in the proof of our result, we have to make a comparison between these two papers. Kato et al. (2015) states that there is no feasible mechanism which is strategy-proof, symmetric and budget balanced. In their model, the indivisible good remaining unallocated is not allowed which is possible in our work. As a result, obviously, leaving the good unallocated forever is possible, however, unattractive. To rule out such mechanism, we impose no wastage which is not used in Kato et al. (2015)

[1] There are a lot of prominent research focusing on Groves mechanisms, such as, Guo and Conitzer (2009), Moulin (2009), Pápai (2003), Ohseto (2006) and Svensson (1983).

[2] Athanasiou (2013) also studies Pareto-optimality within strategy-proof and anonymous mechanisms in the same model.

The rest of this paper is organized as follows. In Section 2, we introduce the model and axioms. In Section 3, we establish the impossibility result. In Section 4, we show independence. In Section 5, we conclude.

2 Notation and definitions

Let $N = \{1, \ldots, n\}$ ($n \in \{2, 3\}$) be the set of agents. Each agent i has a valuation $v_i \in \mathbb{R}_{++}$ of the indivisible good. A valuation profile is a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n_{++}$. Let s_i be agent i's consumption of the indivisible good, and t_i be the monetary transfer of agent i. The consumption bundle of agent i is denoted by $(s_i, t_i) \in \{0, 1\} \times \mathbb{R}$, and the set of *feasible allocation* is $Z = \{(z_1, \ldots, z_n) = ((s_1, t_1), ..., (s_n, t_n)) \in \{\{0, 1\} \times \mathbb{R}\}^n : \sum_{i \in N} s_i \leq 1 \text{ and } \sum_{i \in N} t_i \leq 0\}$. Denote by $C(v) = \{i \in N : s_i(v) = 1\}$ the set of the consumer, and by $NC(v) = \{i \in N : s_i(v) = 0\}$ the set of non-consumers. A *mechanism* is a function $f : \mathbb{R}^n_{++} \to Z$, and given a profile of valuations $v \in \mathbb{R}^n_{++}$, we write $f_i(v) = (s_i(v), t_i(v))$ for all $i \in N$.

We also assume that the preference of each agent i is quasilinear, then his utility function could be represented by $U((s_i, t_i); v_i) = v_i s_i + t_i$. Instead of $(v'_i, v_{-\{i\}})$, we often write (v'_i, v_{-i}) for simplicity which denotes the valuation profile obtained from v by replacing the valuation v_i with v'_i.

Next we introduce several axioms imposed on the feasible mechanisms. The first one is *strategy-proofness* which states that reporting true valuation of the indivisible good is a weakly dominant strategy.

Strategy-proofness: for all $v \in \mathbb{R}^n_{++}$, all $i \in N$, and all $v'_i \in \mathbb{R}_{++}$,

$$U(f_i(v); v_i) \geq U(f_i(v'_i, v_{-i}); v_i).$$

The second one is *anonymity*, which requires the agents to be treated fairly.[3] Let $\pi : N \to N$ be a permutation of N, and Π be the set of all permutations. The valuation profile πv is defined by $(\pi v)_{\pi(i)} = v_i$.

Anonymity: for all $v \in \mathbb{R}^n_{++}$, all $i \in N$, and all $\pi \in \Pi$,

$$v_i s_{\pi(i)}(\pi v) + t_{\pi(i)}(\pi v) = v_i s_i(v) + t_i(v).$$

Third, we introduce *budget balance* which restricts the total amount of monetary transfer to zero.

Budget balance : for all $v \in \mathbb{R}^n_{++}$, $\sum_{i \in N} t_i(v) = 0$.

[3] Ashlagi and Serizawa (2012) employ *anonymity in welfare* as a fairness condition which states that when the valuation of two agents are swapped, their welfare positions are also swapped. Obviously, it is equivalent to anonymity.

At last, we introduce *no wastage*, that is, leave the good unallocated forever is wasteful.

No wastage : for some $v \in \mathbb{R}^n_{++}$, $s_i(v) = 1$ for some $i \in N$.

Remark. *Symmetry* is not used in our characterization which is one of the main axioms in Kato et al. (2015). It requires that agents who report the same valuation get the indifferent consumption bundles. Since Kato et al. (2015)' s result plays an important role in our proof, we prove that anonymity implies symmetry in the next section.

Symmetry: for all $v \in \mathbb{R}^n_{++}$, and all $i, j \in N$,

$$\text{if } v_i = v_j, \text{ then } U(f_i(v); v_i) = U(f_j(v); v_i).$$

3 Main result

Before stating the main result, we present some useful lemmas. In Lemma 1, we consider if the consumption bundle will change by manipulating an agent's valuation in special cases under strategy-prooness.

Lemma 1. *Let f be a strategy-proof mechanism.*
(i) For all $v \in \mathbb{R}^n_{++}$ all $i \in N$, and all $v'_i \in \mathbb{R}_{++}$, if $i \in C(v)$ and $v_i < v'_i$, then $f_i(v) = f_i(v'_i, v_{-i})$.
(ii) For all $v \in \mathbb{R}^n_{++}$ all $i \in N$, and all $v'_i \in \mathbb{R}_{++}$, if $i \in NC(v)$ and $v_i > v'_i$, then $f_i(v) = f_i(v'_i, v_{-i})$.

Proof. The proof is offered in Kato et al. (2015). We omit the details. $\square$

Lemma 2. *A mechanism f is strategy-proof and anonymous if and only if there exist two symmetric function $p : \mathbb{R}^{n-1}_{++} \to \mathbb{R}_+$ and $g : \mathbb{R}^{n-1}_{++} \to \mathbb{R}$ such that, for all $v \in \mathbb{R}^n_{++}$,*

$$f_i(v) = \begin{cases} (1, g(v_{-i}) - p(v_{-i})) & \text{if } v_i > p(v_{-i}) \\ (0, g(v_{-i})) & \text{if } v_i < p(v_{-i}) \end{cases}$$

and

$$f_i(v) \in \{(0, g(v_{-i})), (1, g(v_{-i}) - p(v_{-i}))\} \text{ if } v_i = p(v_{-i}).$$

Moreover, $p(v_{-i}) \geq \max\ v_{-i}$ for all $v_{-i} \in \mathbb{R}^{n-1}_{++}$ and all $i \in N$ where $\max\ v_{-i}$ denotes the maximum of the components of v_{-i}.

Proof. The proof of lemma 2 is offered in Nisan (2007) and Sprumont (2013). We omit the details. $\square$

In the first statement of lemma 2, it says that the mechanism which satisfies strategy-proofness and anonymity is not unique, on the contrary there is a class of strategy-proof and anonymous mechanisms. In the second statement, it studies the feasibility of the mechanism, and says that if there is some agent who consumes the indivisible good, then he must be the agent who reports the highest valuation of the good.

We have a corollary to lemma 2. It follows from that the function g is symmetric.

Corollary 1. *If $v_i = v_j$ for all $i, j \in NC(v)$, then $t_i(v) = t_j(v)$.*

In the next lemma, we show the consumption bundle of each agent at some special profile of valuations, when f is an anonymous and budget balanced feasible mechanism.

Lemma 3. *Let f be an anonymous and budget balanced mechanism. For all $a \in \mathbb{R}_{++}$, let $v = (a, ..., a) \in \mathbb{R}^n_{++}$. If $\sum_{i \in N} s_i(v) = 0$, then $f_i(v) = (0, 0)$ for all $i \in N$. If $\sum_{i \in N} s_i(v) = 1$, then $f_i(v) = (1, -(n-1)a/n)$ for $i \in C(v)$ and $f_j(v) = (0, a/n)$ for all $j \in NC(v)$.*

Proof. We first prove that anonymity implies symmetry. Suppose that f is an anonymous mechanism. Consider a permutation $\pi \in \Pi$, that $\pi(i) = j$, $\pi(j) = i$, and $\pi(k) = k$ for some $i, j \in N$, and all $k \in N \setminus \{i, j\}$. Suppose that $v_i = v_j$. By anonymity, $v_i s_i(v) + t_i(v) = v_i s_{\pi(i)}(\pi v) + t_{\pi(i)}(\pi v)$. By the definition of π, $s_{\pi(i)}(\pi v) = s_j(v)$ and $t_{\pi(i)}(\pi v) = t_j(v)$. Then $v_i s_{\pi(i)}(\pi v) + t_{\pi(i)}(\pi v) = v_i s_j(v) + t_j(v)$. Thus $v_i s_i(v) + t_i(v) = v_i s_j(v) + t_j(v)$. Therefore the mechanism f satisfies symmetry. Next we prove our lemma 3. If $\sum_{i \in N} s_i(v) = 0$, then by corollary 1 $v_i = v_j$ for all $i,j \in N$. By budget balance and, $nt_i = 0$. Thus $t_i = 0$. If $\sum_{i \in N} s_i(v) = 1$, then by symmetry $U(f_i(v); a) = a + t_i(v) = t_j(v) = U(f_i(v); a)$ for $i \in C(v)$ and all $j \in NC(v)$. By budget balance and corollary 1, $t_i(v) + (n-1)t_j(v) = 0$. By solving these two equations, $t_i(v) = -(n-1)a/n$ and $t_j(v) = a/n$. $\square$

Consider another special profile of valuations. In lemma 4, we show the consumption bundle of each agent at this special profile, when f is a strategy-proof, anonymous and budget balanced mechanism.

Lemma 4. *Let f be a strategy-proof, anonymous and budget balanced mechanism. For all $a, b \in \mathbb{R}_{++}$ and $a > b$, let $v = (b, ..., b, a, b, ..., b) \in \mathbb{R}^n_{++}$ such that $v_i = a$ for some $i \in N$, and $v_j = b$ for all $j \in N \setminus \{i\}$. (i) If $s_i(v) = 0$, then $f_i(v) = f_j(v) = (0, 0)$. (ii) If $t_j(v) = 0$, then $f_i(v) = (0, 0)$.*

Proof. *(i)* From lemma 2, $s_j(v) = 0$ for all $j \in N \setminus \{i\}$. By corollary 1, suppose that $t_j(v) = t$, $t \in \mathbb{R}$. By budget balance, $t_i(v) = -(n-1)t$. Consider the profile $v' = (b, ..., b) \in V$ that $v'_i = b$ and $v'_j = v_j$ for all $j \in N \setminus \{i\}$. By lemma 1, $f_i(v') = f_i(v) = (0, -(n-1)t)$. And by lemma 3, $-(n-1)t = b/n$ or 0. If $-(n-1)t = b/n$, then $t = -b/n(n-1)$. By lemma 3, there exists some $k \in N \setminus \{i\}$ that $f_k(v') = (1, -(n-1)b/n)$. Consider the profile $v'' = (b, ..., b, a, b, ..., b) \in V$ that $v''_k = a$ and $v''_j = v'_j$ for all $j \in N \setminus \{k\}$. By lemma 1, $f_k(v'') = (1, -(n-1)b/n)$. Compare the two profiles v and v''. By anonymity, $b/n = a - (n-1)b/n$, which implies $a = b$, a contradiction. Thus the only possibility is $t = 0$ which implies that $f_i(v) = f_j(v) = (0, 0)$. *(ii)* By budget balance,

$t_i(v) = 0$. Suppose that $s_i = 1$. Consider the profile $v' = (b, ..., b) \in V$ that $v'_i = b$ and $v'_j = v_j$ for all $j \in N \setminus \{i\}$. By lemma 3, $f_i(v') = (0, 0)$, $(0, b/n)$ or $(1, -(n-1)b/n)$. However, all of the three violate strategy-proofness. Therefore *(ii)* holds. $\square$

In the next part of this section, we introduce our main result formally and show the proof.

Theorem. *There is no feasible mechanism which is strategy-proof, anonymous, budget balanced and no wasteful.*

We briefly explain how to prove it. First, we show that there must be some profile $v^x = (x, ..., x) \in \mathbb{R}^n_{++}$ such that $f_i(v) = (0, 0)$ for all $i \in N$. Then for all $v = (v_1, ..., v_n)$ $v_i \in (0, x)$ $i \in N$, we have $f_i(v) = (0, 0)$. Third, we define the supremum of x as α. At last, we consider the profiles of each agent's valuation bigger than α. At all these kind of profiles, there must be some agent who consumes the good which is contradictory to the impossibility result proved in Kato et al. (2015).

Lemma 5. *Let f be a strategy-proof, anonymous, budget balanced and no wasteful mechanism. There exists some profile $v^x = (x, ..., x) \in \mathbb{R}^n_{++}$ for some $x \in \mathbb{R}_{++}$ such that $f_i(v^x) = (0, 0)$ for all $i \in N$.*

Proof. First, we state that the indivisible good must be unallocated at some profile of valuations. If not, then the good must be allocated at all profile of valuations which means that there is some strategy-proof, anonymous and budget balanced mechanism. This is contradictory to the impossibility result proved in Kato et al. (2015), since anonymity implies symmetry. Second, from any given profile at which the indivisible good is unallocated, we can find a profile which proves lemma 5. Since the number of agents is two or three, we distinguish two cases.

Case 1. $n = 2$.

Suppose that $\sum_{i \in N} s_i(v) = 0$, $v = (a, b) \in \mathbb{R}^n_{++}$ for some $a, b \in \mathbb{R}_{++}$. Let $t_1(v) = t$, $t \in \mathbb{R}$, by budget balance, then $t_2(v) = -t$. There are two subcases. In subcase 1, $a = b$, and without loss of generality we suppose that $a > b$ in subcase 2.

Subcase 1. $a = b$.

By Lemma 3, $f_1(v) = f_2(v) = (0, 0)$. Therefore Lemma 5 holds.

Subcase 2. $a > b$.

By Lemma 4, $t = 0$. Consider the profile $v' = (b, b)$. By Lemma 1, $f_1(v') = (0, 0)$. By Lemma 3, $f_1(v') = f_2(v') = (0, 0)$ which proves Lemma 5.

Case 2. $n = 3$.

Suppose that $\sum_{i \in N} s_i(v) = 0$, $v = (a, b, c) \in \mathbb{R}^n_{++}$ for some a, b and $c \in \mathbb{R}_{++}$. Let $t_1(v) = \beta$, $t_2(v) = \gamma$ and $t_3(v) = \delta$ for some β, γ and $\delta \in \mathbb{R}$. By budget balance, $\beta + \gamma + \delta = 0$. We analyze this problem by distinguishing four subcases. In subcase 1, $a = b = c$. Without loss of generality, we assume that $a > b > c$ in subcase 2, $a = b > c$ in subcase 3 and $a > b = c$ in subcase 4.

Subcase 1. $a = b = c$.

By Lemma 3, $f_1(v) = f_2(v) = f_3(v) = (0, 0)$. Therefore Lemma 5 holds.

Subcase 2. $a > b > c$.

Consider the profile $v^1 = (b, b, c)$. By Lemma 1, $f_1(v^1) = (0, \beta)$. By anonymity, $f_2(v^1) = (1, \beta - b)$ or $(0, \beta)$. Suppose that $f_2(v^1) = (1, \beta - b)$. Consider the profile $v^2 = (b, a, c)$. Comparing profile v with profile v^2, by anonymity we have $\beta = a + \beta - b$, meaning that $a = b$, a contradiction. Thus $f_2(v^1)$ must be $(0, \beta)$, and by Lemma 2 and budget balance $f_3(v^1) = (0, -2\beta)$.

Consider the profile $v^3 = (b, b, b)$. By Lemma 3, $f_3(v^3) = (0, b/3)$, $(1, -2b/3)$ or $(0, 0)$.

(1). If $f_3(v^3) = (0, b/3)$. By Lemma 1, $-2\beta = b/3$, meaning that $\beta = -b/6$.

(2). If $f_3(v^3) = (1, -2b/3)$. This implies that $f_1(v^3) = f_2(v^3) = (0, b/3)$. Consider the profile $v^4 = (c, b, b)$. By Lemma 1, $f_1(v^4) = (0, b/3)$. Comparing profile v^4 with profile v^1, by anonymity we have $-2\beta = b/3$, meaning that $\beta = -b/6$.

From (1) and (2), we know that $\beta = -b/6$. Now we consider the profile $v^5 = (c, b, c)$. By Lemma 1, $f_1(v^5) = (0, -b/6)$. By Lemma 2 and Corollary 1, $f_3(v^5) = (0, -b/6)$. Then by Lemma 2 and budget balance, we have $f_2(v^5) = (1, b/3)$ or $(0, b/3)$. Consider the profile $v^6 = (c, c, c)$. If $f_2(v^5) = (1, b/3)$, then $f_2(v^6)$ can not be $(0, 0)$ or $(0, c/3)$ because of strategy-proofness. But if $f_2(v^6) = (1, -2c/3)$, by Lemma 1 we have $b = -2c$, a contradiction. If $f_2(v^5) = (0, b/3)$, then by Lemma 1 we have $f_2(v^6) = (0, b/3)$ which violates Lemma 3.

(3). From the analysis above, the only possibility is that $f_3(v^3) = (0, 0)$. By Lemma 3, $f_1(v^3) = f_2(v^3) = f_3(v^3) = (0, 0)$ which proves Lemma 5.

Subcase 3. $a = b > c$

By anonymity and budget balance, $t_1(v) = t_2(v) = \beta$, $t_3(v) = -2\beta$. Consider the profile $v^1 = (a, a, a)$. By Lemma 3, $f_3(v^1) = (0, a/3)$, $(1, -2a/3)$ or $(0, 0)$.

(1). If $f_3(v^1) = (0, a/3)$. By Lemma 1, $-2\beta = a/3$, meaning that $\beta = -a/6$.

(2). If $f_3(v^1) = (1, -2a/3)$. This implies that $f_1(v^1) = f_2(v^1) = (0, a/3)$. Consider the profile $v^2 = (c, a, a)$. By Lemma 1, $f_1(v^2) = (0, a/3)$. Comparing profile v^2 with profile v^1, by anonymity we have $-2\beta = a/3$, meaning that $\beta = -a/6$.

From (1) and (2), we know that $\beta = -a/6$. Now we consider the profile $v^3 = (c, a, c)$. By Lemma 1, $f_1(v^3) = (0, -a/6)$. By Lemma 2 and Corollary 1, $f_3(v^3) = (0, -a/6)$. Then by Lemma 2 and budget balance, we have $f_2(v^3) = (1, a/3)$ or $(0, a/3)$. Consider the profile $v^4 = (c, c, c)$. If $f_2(v^3) = (1, a/3)$, then $f_2(v^4)$ can not be $(0, 0)$ or $(0, c/3)$ because of strategy-proofness. But if $f_2(v^4) = (1, -2c/3)$, by Lemma 1 we have $a = -2c$, a contradiction. If $f_2(v^3) = (0, a/3)$, then by Lemma 1 we have $f_2(v^4) = (0, a/3)$ which violates Lemma 3.

(3). From the analysis above, the only possibility is that $f_3(v^1) = (0, 0)$. By Lemma 3, $f_1(v^1) = f_2(v^1) = f_3(v^1) = (0, 0)$ which proves Lemma 5.

Subcase 4. $a > b = c$

By Lemma 4, $f_1(v) = f_2(v) = f_3(v) = (0, 0)$. Consider the profile $v^1 = (b, b, b)$. By Lemma 1 and 3, $f_1(v^1) = f_2(v^1) = f_3(v^1) = (0, 0)$ which proves Lemma 5. $\square$

Lemma 5 states the existence of $v^x = (x, ..., x) \in \mathbb{R}^n_{++}$ for some $x \in \mathbb{R}_{++}$ such that $f_i(v^x) = (0, 0)$ for all $i \in N$. Next we turn to consider what is the consumption bundle of each agent at the profile $v = (v_1, ..., v_n) \in \mathbb{R}^n_{++}$ such that $v_i < x$ for all $i \in N$.

Lemma 6. *Let f be a strategy-proof, anonymous, budget balanced and no wasteful mechanism. For some $x \in \mathbb{R}_{++}$, let $v^x = (x, ..., x) \in \mathbb{R}^n_{++}$ such that $f_i(v^x) = (0, 0)$ for all $i \in N$. For all $y \in (0, x)$, let $v^y = (y, ..., y) \in V$. Then $f_i(v^y) = (0, 0)$ for all $i \in N$, where $n = 3$.*

Proof. Consider the profile $v^1 = (x, x, y)$. By Lemma 1, $f_3(v^1) = (0, 0)$. We distinguish two possible cases, namely, $\sum_{i \in N} s_i(v^1) = 0$ and $\sum_{i \in N} s_i(v^1) = 1$.

Case 1. $\sum_{i \in N} s_i(v^1) = 0$.

By anonymity and budget balance, $f_1(v^1) = f_2(v^1) = (0, 0)$. Consider the profile $v^2 = (x, y, y)$. By Lemma 1, $f_2(v^2) = (0, 0)$. By Lemma 2 and Corollary 1, $f_3(v^2) = f_2(v^2) = (0, 0)$. Then by Lemma 4, $f_1(v^2) = (0, 0)$. Consider the profile $v^3 = (y, y, y)$. By Lemma 1, $f_1(v^3) = (0, 0)$, and by Lemma 3 $f_1(v^3) = f_2(v^3) = f_3(v^3) = (0, 0)$ which proves Lemma 6.

Case 2. $\sum_{i \in N} s_i(v^1) = 1$.

Without loss of generality, assume $s_1(v^1) = 1$. By anonymity and budget balance, $f_1(v^1) = (1, -x/2)$ and $f_2(v^1) = (0, x/2)$. Consider the profile $v^2 = (x, y, y)$. By Lemma 1, $f_2(v^2) = (0, x/2)$. By Lemma 2 and Corollary 1, $f_3(v^2) = f_2(v^2) = (0, x/2)$. Then by Lemma 2 and budget balance, $f_1(v^2) = (0, -x)$ or $(1, -x)$. We distinguish two subcases.

Subcase 1. $f_1(v^2) = (0, -x)$.

Consider the profile $v^3 = (y, y, y)$. By Lemma 1, $f_1(v^3) = (0, -x)$ which violates Lemma 3.

Subcase 2. $f_1(v^2) = (1, -x)$.

Consider the profile $v^3 = (y, y, y)$. By Lemma 3, $f_1(v^3) = (1, -2y/3)$, $(0, y/3)$ or $(0, 0)$. If $f_1(v^3) = (1, -2y/3)$, by Lemma 1 we have $x = 2y/3$, a contradiction. If $f_1(v^3) = (0, y/3)$, by strategy-proofness we have $y \leq 0$, a contradiction. The only possibility is that $f_1(v^3) = (0, 0)$, and by Lemma 3 we have $f_1(v^3) = f_2(v^3) = f_3(v^3) = (0, 0)$ which proves Lemma 6. $\square$

In the next lemma, we show that if each component of a valuation profile is less than x, then the strategy-proof, anonymous, budget balanced and no wasteful mechanism leaves the indivisible good unallocated and makes no monetary transfer at this profile.

Lemma 7. *Let f be a strategy-proof, anonymous budget balanced and no wasteful mechanism. For some $x \in \mathbb{R}_{++}$, let $v^x = (x, ..., x) \in \mathbb{R}^n_{++}$such that $f_i(v^x) = (0, 0)$ for all $i \in N$. Let $v = (v_1, ..., v_n) \in \mathbb{R}^n_{++}$* and *$v_i \in (0, x)$ for all $i \in N$. Then $f_i(v) = (0, 0)$ for all $i \in N$.*

Proof. Since the number of agents is two or three, we distinguish two cases, namely, $n = 2$ and $n = 3$.

Case 1. $n = 2$.

Consider the profile $v^1 = (x, b)$ for all $b \in (0, x)$. By Lemma 1, $f_2(v^1) = (0, 0)$. By Lemma 4, $f_1(v^1) = f_2(v^1) = (0, 0)$. Consider the profile $v^2 = (a, b)$ for all $a \in (0, x)$. By Lemma 1, $f_1(v^2) = (0, 0)$. If $a = b$, then by Lemma 3 we have $f_1(v^2) = f_2(v^2) = (0, 0)$. If $a > b$ or $a < b$, by Lemma 4 we have $f_1(v^2) = f_2(v^2) = (0, 0)$. Therefore, Lemma 7 holds.

Case 2. $n = 3$.

We distinguish three patterns of the valuation profiles, namely, $v = (a, a, b)$, (a, b, b) and (a, b, c) for all a, b and $c \in (0, x)$. Without loss of generality, we assume that $a > b > c$.

Subcase 1. $v = (a, a, b)$.

By Lemma 1 and 6, $f_3(v) = (0, 0)$. If $\sum_{i \in N} s_i(v) = 0$, then $f_1(v) = f_2(v) = (0, 0)$. Therefore Lemma 7 holds.

Suppose not. Without loss of generality, assume that $s_1(v) = 1$. By anonymity and budget balance, $f_1(v) = (1, -a/2)$ and $f_2(v) = (0, a/2)$. Consider the profile $v^1 = (a, b, b)$. By Lemma 1, $f_2(v^1) = f_2(v) = (0, a/2)$. By Lemma 2 and Corollary 1, $f_2(v^1) = f_3(v^1) = (0, a/2)$. By Lemma 4 and budget balance, $f_1(v^1) = (1, -a)$. Consider the profile $v^2 = (d, b, b)$ for all $d \in (a, x)$. By Lemma 1, $f_1(v^2) = (1, -a)$. By budget balance and Corollary 1, $f_2(v^2) = f_3(v^2) = (0, a/2)$. Consider the profile $v^3 = (d, d, b)$. By Lemma 1 and 6, $f_3(v^3) = (0, 0)$. By anonymity and budget balance, $f_2(v^3) = (0, 0)$, $(0, d/2)$ or $(1, -d/2)$. However, $f_2(v^3) = (0, 0)$ which implies $a = 0$ or $(0, d/2)$ which implies $a = d/2$ are impossible. Thus $f_2(v^3) = (1, -d/2)$. By anonymity, $f_2(v^3) = (0, d/2)$. Consider the profile $v^4 = (b, d, b)$. By Lemma 1, $f_1(v^4) = (0, d/2)$. By Lemma 2 and Corollary 1, $f_1(v^4) = f_3(v^4) = (0, d/2)$. By Lemma 5, $f_2(v^4) = (1, -d)$. Comparing profile v^2 with profile v^4, by anonymity we have $d - a = d - d$, meaning that $a = d$, a contradiction.

Subcase 2. $v = (a, b, b)$.

Consider the profile $v^1 = (a, a, b)$. From subcase 1 proved above, we have $f_1(v^1) = f_1(v^1) = f_1(v^1) = (0, 0)$. By Lemma 1, $f_2(v) = f_2(v^1) = (0, 0)$. By Lemma 2 and Corollary 1, $f_2(v) = f_3(v) = (0, 0)$. By Lemma 4, $f_1(v) = (0, 0)$. Therefore Lemma 7 holds.

Subcase 3. $v = (a, b, c)$.

Consider the profile $v^1 = (a, b, b)$. From subcase 2 proved above, we have $f_3(v^1) = (0, 0)$. By Lemma 1, $f_3(v) = f_3(v^1) = (0, 0)$. Consider the profile $v^2 = (a, a, c)$. From subcase 1, we have

$f_2(v^2) = (0,0)$. By Lemma 1, $f_2(v) = f_2(v^2) = (0,0)$. By Lemma 2 and budget balance, $f_1(v) = (1,0)$ or $(0,0)$. Consider the profile $v^3 = (b, b, c)$. From subcase 1, we have $f_1(v^3) = (0,0)$. Thus $f_1(v)$ can not be (1,0) which violates strategy-proofness. Therefore $f_1(v) = f_2(v) = f_2(v) = (0,0)$ which proves Lemma 7. $\square$
No wastage sets a boundary of x. We prove this in the next lemma.

Lemma 8. *Let f be a strategy-proof, anonymous, budget balanced and no wasteful mechanism. For some $x \in \mathbb{R}_{++}$, let $v^x = (x, ..., x) \in \mathbb{R}^n_{++}$ such that $f_i(v^x) = (0,0)$ for all $i \in N$. Define $\alpha = sup$ x. Then $0 < \alpha < +\infty$.*

Proof. Since $x \in \mathbb{R}_{++}$, we have $\alpha > 0$. Suppose $\alpha = +\infty$. Then $f_i(v^x) = (0,0)$, $v^x = (x, ..., x) \in \mathbb{R}^n_{++}$ for all $x \in \mathbb{R}_{++}$ and all $i \in N$. By Lemma 7, for all $v = (v_1, ..., v_n) \in \mathbb{R}^n_{++}$ and all $i \in N$ which is contradictory to no wastage. Therefore $\alpha < +\infty$. $\square$

Proof of Theorem. Let $v_i \in (\alpha, +\infty)$ for all $i \in N$. Then for all $v = (v_1, ..., v_n) \in \mathbb{R}^n_{++}$, we have $\sum_{i \in N} s_i(v) = 1$. If not, by Lemma 5 there must be some $\alpha+\epsilon$ such that $v^{\alpha+\epsilon} = (\alpha+\epsilon, ..., \alpha+\epsilon) \in \mathbb{R}^n_{++}$ and $f_i(v^{\alpha+\epsilon}) = (0,0)$ for all $i \in N$ which is contradictory to $\alpha = sup\ x$. This statement is contradictory to the impossibility result proved in Kato et al. (2015). Therefore there is no feasible mechanism which is strategy-proof, anonymous, budget balanced and no wasteful. $\square$

4 Independence

At last we show the independence of axioms in Theorem 1. We introduce a notation: for all $v = (v_1, ..., v_n) \in \mathbb{R}^n_{++}$, $v^{[1]}$ denotes the highest valuation in v.

Example 1 Fix $\alpha \in \mathbb{R}_{++}$. For all $v = (v_1, ..., v_n) \in \mathbb{R}^n_{++}$, let f be such that
(1). if $v^{[1]} < \alpha$, then $f_i(v) = (0,0)$ for all $i \in N$.
(2). if $v^{[1]} \geq \alpha$, let $C(v) \subset argmax_{i \in N}\ v_i$. Then $f_i(v) = (1, -(n-1)v^{[1]}/n)$ for $i \in C(v)$ and $f_j(v) = (0, v^{[1]}/n)$ for all $j \in NC(v)$.
Then, f satisfies anonymity, budget balance and no wastage but not strategy-proofness.

Example 2 Fix $\alpha \in \mathbb{R}_{++}$. For all $v = (v_1, ..., v_n) \in \mathbb{R}^n_{++}$, let f be such that
(1). if $v_1 < \alpha$, then $f_i(v) = (0,0)$ for all $i \in N$.
(2). if $v_1 \geq \alpha$, then $f_1(v) = (1, -v_1)$ and $f_i(v) = (0, v_1/(n-1))$ for all $i \in N \setminus \{i\}$.
Then, f satisfies strategy-proofness, budget balance and no wastage but not anonymity.

Example 3 (Sprumont 2013). Let f be the mechanism stated in lemma 2 and set $p(v_{-i}) = \max(\max\ v_{-i}, \alpha)$ and $g(v_{-i}) = g_0(v^{[1]}) = \text{med}(0,\ v^{[1]} - \alpha,\ \alpha/(n-1))$ for some $\alpha \in \mathbb{R}_{++}$.
Then, f satisfies strategy-proofness, anonymity and no wastage but not budget balance.

Example 4 Let $f_i(v) = (0,0)$ for all $v \in \mathbb{R}^n_{++}$, and all $i \in N$.
Then, f satisfies strategy-proofness, anonymity and budget balance but not no wastage.

5 Concluding remarks

In this paper, we show that there is no feasible mechanism which is strategy-proof, anonymous, budget balanced and no wasteful. In our model, we assume that the number of agents is two or three. We do not know if our theorem remains valid when $n \geq 4$. We leave this question for future research.

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