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Revelation principle under strategic uncertainty: application to financial contracts with limited liability

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Abstract

We consider a principal-agent model in which the principal can terminate the agent's project and an outsider can affect the project's result. Only the agent can observe the outsider's action and sends a message to the principal. Under strategic uncertainty about the outsider's action in complete information, sequential equilibrium is a suitable equilibrium concept to select the robust outcome and to completely identify the underlying posterior belief. We prove the revelation principle for sequential equilibrium in such a game. Based on this revelation principle, we present a legitimate and simple form of the limited liability constraint on a financial contract that is robust to strategic uncertainty.

JEL classification: D86, L14, C72

Keywords: revelation principle, sequential equilibrium, strategic uncertainty, limited liability, financial contracts

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1 Introduction

In this paper, we show a version of revelation principle in a special class of complete but imperfect games with a principal, an agent and an "outsider." The agent has a plan of a project and needs the principal's approval (for investment, permit, etc.) to launch and continue it; to get approval, the agent promises to pay to the principal from the future profit. Only the agent can observe the action of the outsider, which affects the agent's profit and liquidity holding. Termination or continuation of the project also affects the outsider's profit. Unlike the standard version of revelation principle,[1] private information is the outsider's action and thus an endogenous variable in the model. We verify that any outcome in a sequential equilibrium under an arbitrary mechanism can be truthfully implemented under a version of a direct mechanism; this is our revelation principle. The "outcome" includes not only the ex-ante unconditional probability with which the project continues but also the interim continuation probability conditional on the outsider's action. Hence the outsider chooses the same strategy.

The motivation comes from theory of financial contracts under product market competition.[2] It is common in the analysis of financial contracts to impose limited liability constraint, which restricts monetary payment to within the borrower's liquidity holding. The liquidity holding is uncertain if the profit is unobservable or unverifiable. Revelation principle allows us to focus on equilibrium in a direct mechanism, where the borrower truthfully and directly tells his private information and consequently his liquidity holding is uniquely inferred from the message. It is common to assume exogenous fundamental uncertainty, e.g., about productivity or success of R&D so we use Bayesian Nash or perfect Bayesian equilibrium as a solution. However, one may want to focus on strategic uncertainty about the rival firm's strategy to shed light on effects of competitive pressure on the entrant's financing. Then, the game is of complete but imperfect information and the agent's private information is generated by the outsider of the contract. This is the difference from the standard version of revelation principle under incomplete information games.

In our model, truthful implementation should induce the agent to truthfully report the outsider's action and the principal to believe it, whatever the outsider's action is—especially even if the reported action of the outsider is out of the equilibrium path of play. We explicitly specify the belief that the agent's message implies about the outsider's action. Applied game theorists may conventionally focus on pure-strategy equilibria; then only one action is realized in equilibrium play. But we need to determine all the off-path belief. So we adopt the concept of sequential equilibrium: we introduce perturbation to strategies, which enables us to wholly determine the belief by Bayes' law. An equilibrium configuration of strategies and belief is defined as the limit of a sequence of perturbed equilibria. The perturbation can be seen as representing strategic uncertainty. Even though the players play and believe in the

[1] For the formal statement and the proof, see Fudenberg and Tirole (1991, Section 7.2) for instance. In the context of mechanism design, see Diamantaras et al. (2009) for the revelation principles for dominant strategy equilibrium and Baysian Nash equilibrium. Bester and Strausz (2001) prove a version for perfect Bayesian equilibrium with renegotiation of a contract.

[2] For example, Bolton and Scharfstein (1990), Aghion et al. (2000), Poitevin (1989). They analyze the relationship between financial contracts and competitive pressure in the framework of Bayesian games: they embed uncertainty in the economy's fundamental such as demand and cost functions and solve perfect Bayesian Nash equilibrium. See Tirole (2006, Section 7.1) for the survey of such models.

strategy profile in the sequential equilibrium, they fear possibility that the opponents could deviate from it. Sequential equilibrium imposes robustness to such strategic uncertainty on the equilibrium play.

Based on our version of revelation principle, we can extend the limited liability constraint to our situation. As long as a message allows continuation of the project—even if this message is never sent in the equilibrium play, the future profit and thus the liquidity holding are uniquely inferred from the message. The limited liability constraint restricts the repayment promised by such a message to within the liquidity holding *inferred from this message*. This can be interpreted as if the lender and the borrower realize a possibility that the outsider could take a non-equilibrium strategy and make the promised repayment still payable even if the outsider deviates from the equilibrium strategy. In short, our version of limited liability constraint makes the contract robustly feasible under strategic uncertainty.

In the analysis of effects of product market competition on financial contracts, the core of the analysis would be the condition for the borrower to successfully launch and continue his business. In our terms, the most interesting case would be an equilibrium play in which the agent continues the project. Precisely our truthful implementation requires truthful and direct announcement only when the project continues, leaving his announcement unchanged from the original equilibrium (possibly not truth telling) in the cases of termination. Besides, even after applying the revelation principle, we may have multiple equilibria under our version of the direct mechanism. But the revelation principle helps us to induce necessary conditions for continuation of the project.

To show an example of application, we consider a financial contract of an entrant under threat of predation. To enter the market, the entrant needs to borrow fixed costs from an investor, who cannot observe the intensity of predatory conduct by an incumbent. The predation reduces both the incumbent's and the entrant's profits. However, by lowering the entrant's profitability, it may deter the entrant's entry. From our revelation principle and limited liability constraint, we can easily find that a positive equilibrium profit is *not sufficient* for the entrant to borrow entry costs.

About complete information games, one might think of the revelation principle for correlated equilibrium.[3] In terms of contract theory, this version can be interpreted as if the principal coordinates multiple agents' actions according to some signaling procedure and the revelation principle allows us to reduce the domain of such signaling to a direct mechanism in which each agent directly reports one's own action plan. In this version, the principal directly coordinates actions over *all the agents*. In our situation, the outsider's action is not directly controlled by the principal, and *the outsider* does not send any message to the principal.

Gerardi and Myerson (2007) employ sequential equilibrium to analyze a Bayesian communication game with a non-full support on the type space. In their model, the agents' *actions* are perturbed while the type distribution is fixed. One might interpret the outsider's action in our model as the "type" of an agent in their communication game. Yet, their perturbation is essentially different from ours, since we want to perturb the outsider's action (the agent's *type* in this interpretation). Besides, while every player sends one's own message in their commu-

[3] See Osborne and Rubinstein (1994, Proposition 47.1).

nication game, the outsider does not send any message in our model. Hence we cannot apply their results to our model.

The paper proceeds as follows. We formally set up the model in the next section and characterize a sequential equilibrium under an arbitrary mechanism in Section 3. Our version of revelation principle is verbally explained in Section 4, while a formal presentation is given with the proof in Appendix. Based on this revelation principle, we propose the limited liability constraint under strategic uncertainty in Section 5. In Section 6, we apply the revelation principle to a financial contract of an entrant under threat of predation by an incumbent. The last section concludes the paper.

2 The model

We imagine a situation in which the principal decides on whether or not the agent can continue a project whose outcome is affected by an outsider. More specifically, the outsider chooses action a from set $\mathcal{A}$ in period 1. In period 2, the principal decides on whether to continue the project or quit it. Finally the agent pays $d \in \mathbb{R}$ to the principal. The principal cannot observe the outsider's action, while the agent can.

To induce information, the principal and the agent use a mechanism. A mechanism consists of message space M, interim continuation schedule $C : M \to \{0, 1\}$, and payment schedule $D : M \to \mathbb{R}$. After observing the outsider's action, the agent sends message $m \in M$ to the principal. Depending on m, the principal chooses continuation $C(m) = 1$ or termination $C(m) = 0$ of the project. Then, the agent pays $D(m)$ to the principal.

When the outsider's action is $a \in \mathcal{A}$ and the promised payment is $d \in \mathbb{R}$, The payoffs are $u_1(a, d)$ for the agent, $v_1(a)$ for the outsider and $w_1(a, d)$ for the principal if the project continues; if it terminates, they are $u_0(d), v_0(a)$ and $w_0(a, d)$, respectively. We assume that the outsider's action is irrelevant to the agent's payoff if the project terminates.

We assume that there are only finitely many, but at least two, feasible actions of the outsider. Similarly we limit our attention to the mechanism in which the agent has only finitely many, but at least two, available messages.

Example 1 (Financial contract under threat of predation). Consider a financial contract between a new entrepreneur (the agent) and an investor (the principal). The entrepreneur is planning to start a new business but needs the investor to pay for the entry cost $F > 0$. The incumbent (the outsider) responds to the entrepreneur's entry plan by predatory conducts such as capacity expansion, excessive advertisement, or price cut. Denote by a the intensity of such conducts and $\mathcal{A} \subset \mathbb{R}$ be the set of feasible degrees. d is the repayment from the entrant to the investor and can depend on the message.

Let $U_1(a)$ and $V_1(a)$ be the operating profits of the entrant and of the incumbent when the degree of predation is a and the entrant still continues his business. Let $V_0(a)$ be that of the incumbent when the entrant exists from the market, while the entrant's operating profit is zero in this case. We assume that $U_1 : \mathcal{A} \to \mathbb{R}$ is a decreasing function and $V_0, V_1 : \mathcal{A} \to \mathbb{R}$ are concave functions of a, and $V_0(a) > V_1(a)$ for each $a \in \mathcal{A}$. They are common knowledge and their values become certain once a is determined.[4] But, a is not observable or verifiable to the investor; so the investor has to rely on the entrepreneur's voluntary report (the message) about it.

In sum, the payoffs are $u_1(a,d) = U_1(a) - d$ for the entrant, $w_1(a,d) = d - F$ for the investor, and $v_1(a) = V_1(a)$ for the incumbent, if the entrant enters the market. If he quits, they are $u_0(d) = -d$, $w_0(a,d) = d$ and $v_0(a) = V_0(a)$, respectively.

Later we impose the limited liability constraint on the financial contract: repayment d should be within the anticipated liquidity holding. Suppose that the entrepreneur has no liquidity before starting the business. So $U_1(a)$ is his liquidity holding if the entrant enters; it is zero if the entry plan terminates.

3 Characterization of a sequential equilibrium

Given mechanism (M, C, D), the agent's strategy is which message in M to send after observing the outsider's action in $\mathcal{A}$. The outsider's strategy is the choice of an action from $\mathcal{A}$. We define the space of mixed (behavioral) strategies as follows.[5]

$$\begin{aligned} \text{The outsider's action strategy:} \quad & \sigma_{\mathcal{A}} \in \Delta\mathcal{A}. \\ \text{The agent's messaging strategy conditional on } a \in \mathcal{A}\text{:} \quad & \sigma_M(\cdot|a) \in \Delta M. \end{aligned}$$

That is, the outsider takes action a with probability $\sigma_{\mathcal{A}}(a)$ and then the agent sends message m with probablity $\sigma_M(m|a)$. Let Σ be the space of feasible strategy profiles $\sigma = (\sigma_{\mathcal{A}}, \sigma_M)$ and $\mathring{\Sigma}$ the set of completely mixed strategy profiles. Posterior belief $\mu(\cdot|m) \in \Delta\mathcal{A}$ is a probability measure on $\mathcal{A}$, conditional on $m \in M$; receiving message m, the principal believes that the outsider chooses action a with probability $\mu(a|m)$. Below we define and characterize a sequential equilibrium (σ^*, μ^*).

Definition 1 (Sequential equilibrium)**.** Given a mechanism (M, C, D), the pair of a mixed strategy profile $\sigma^* \in \Sigma$ and posterior belief $\mu^* \in \Delta\mathcal{A} \times M$ is a **sequential equilibrium** if there is a sequence of completely mixed strategy profiles $\{\sigma^k\}_{k\in\mathbb{N}} \subset \mathring{\Sigma}$ that converges to σ^* and satisfies the following properties.

i) (Consistency of belief) μ^* is the limit of Bayesian beliefs $\{\mu^k\}_{k\in\mathbb{R}}$ induced from $\{\sigma^k\}_{k\in\mathbb{N}}$: for each $a \in \mathcal{A}$ and $m \in M$, μ^* satisfies

$$\mu^k(a|m) := \frac{\sigma^k_M(m|a)\sigma^k_{\mathcal{A}}(a)}{\sum_{a'\in\mathcal{A}} \sigma^k_M(m|a')\sigma^k_{\mathcal{A}}(a)} \longrightarrow \mu^*(a|m) \qquad \text{as } k \to \infty.$$

[4] This implies that we neglect possible fundamental uncertainty about the payoff *functions* and focus on strategic uncertainty. This fits well with conventional brick-and-mortar local retailers and small commodity manufacturers such as local grocery stores, gas stations, and non-brand food/apparel manufacturers, rather than venture companies in innovative industries.

[5] Throughout this paper, ΔX denotes the set of probability measures on set X. In particular, if X is a finite set, ΔX is an $\sharp X$-dimensional simplex, i.e., $\Delta X := \{\sigma \in \mathbb{R}^{\sharp X}_{+} | \sum_{x\in X} \sigma(x) = 1\}$. Let $\mathring{\Delta} X := \{\sigma \in \mathbb{R}^{\sharp X}_{++} | \sum_{x\in X} \sigma(x) = 1\}$. Then, $\sigma \in \mathring{\Sigma}$ means $\sigma_{\mathcal{A}} \in \mathring{\Delta}\mathcal{A}$ and $\sigma_M(\cdot|a) \in \mathring{\Delta} M$ for all $a \in \mathcal{A}$.

ii) (Sequential rationality) For each $a \in \mathcal{A}$, $\sigma_M^*(\cdot|a)$ is the agent's optimal messaging strategy such that

$$\begin{aligned}\max_{\sigma_M(\cdot|a)\in\Delta M} \quad & \mathbb{E}_m\left[C(m)u_1(a,D(m)) + (1-C(m))u_0(D(m))|a\right] \\ &= \sum_{m\in M}\{C(m)u_1(a,D(m)) + (1-C(m))u_0(D(m))\}\sigma_M(m|a).\end{aligned}$$

$\sigma_{\mathcal{A}}^*$ is the outsider's optimal strategy such that

$$\begin{aligned}\max_{\sigma_{\mathcal{A}}\in\Delta\mathcal{A}} \quad & \mathbb{E}_{m,a}\left[C(m)v_1(a) + (1-C(m))v_0(a)\right] \\ &= \sum_{a\in\mathcal{A}}\sum_{m\in M}\{C(m)v_1(a) + (1-C(m))v_0(a)\}\,\sigma_M^*(m|a)\sigma_{\mathcal{A}}(a).\end{aligned}$$

Denote by $M_1(a)$ (M_0, resp.) the set of the most desirable messages for the agent among the available messages that let the project continue (terminate, resp.):

$$M_1(a) := \arg\max_{m\in M} u_1(a,D(m)) \text{ s.t. } C(m)=1, \quad M_0 := \arg\max_{m\in M} u_0(D(m)) \text{ s.t. } C(m)=0. \quad (1)$$

In the agent's optimal messaging strategy, as long as message m is sent with some positive probability, $C(m)=1$ ($C(m)=0$, resp.) implies $m \in M_1(a)$ ($m \in M_0$, resp.). Let $\underline{D}_1(a)$ ($\underline{D}_0$, resp.) be the value of $D(m)$ with $m \in M_1(a)$ ($m \in M_0$, resp.).[6]

Given that the interim continuation schedule is binary, i.e., $C(m) \in \{0,1\}$, we can classify all the *possibly* sent messages into two sets:

$$M_c^* := \{m \in M | C(m) = c,\ \exists a \in \mathcal{A}\ \sigma_M^*(m|a) > 0\} \quad \text{for each } c \in \{0,1\}.$$

M_c^* is the set of messages that satisfy $C(m)=c$ and are sent with a positive probability after some a. Given a, let $P^*(a)$ be the equilibrium ex-ante probability of continuation, i.e., the probability of sending messages in M_1^*:

$$P^*(a) := \sum_{m\in M}\sigma_M^*(m|a)C(m) = \sum_{m\in M_1^*}\sigma_M^*(m|a). \quad (2)$$

A_1^* is the set of actions after which the project continues with a positive probability in the equilibrium:

$$A_1^* := \{a \in \mathcal{A} | P^*(a) > 0\}. \quad (3)$$

From the above argument, the outsider's payoff reduces to

$$\sum_{a\in A_1^*}\left\{P^*(a)v_1(a) + (1-P^*(a))v_0(a)\right\}\sigma_{\mathcal{A}}(a) + \sum_{a\notin A_1^*} v_0(a)\sigma_{\mathcal{A}}(a). \quad (4)$$

[6]If multiple values of d correspond to the maximum in (1), then we select any one of them.

The agent's payoff maximization reduces to

$$\max_a u_1(a, \underline{D}_1(a)) \sum_{m \in M_1^*} \sigma_M(m|a) + u_0(\underline{D}_0(a)) \sum_{m \in M_0^*} \sigma_M(m|a).$$

We categorize equilibria by *possibly* sent messages, i.e., the ones in M that are sent with positive probability conditional on some $a \in \mathcal{A}$.

1) a pooling-continuation equilibrium $M_0^* = \emptyset$: the project continues after any a, i.e., $A_1^* = \mathcal{A}$;

2) a pooling-termination equilibrium $M_1^* = \emptyset$: the project terminates after any a, i.e., $A_1^* = \emptyset$;

3) a separating equilibrium $M_0^*, M_1^* \neq \emptyset$: continuation/termination depends on a, i.e., $\emptyset \neq A_1^* \subsetneq \mathcal{A}$.

In a separating equilibrium, the agent sends only messages in the set M_1^* (M_0^*, resp.) with a positive probability and $P^*(a)$ is equal to one (zero, resp.) if $u_1(a, \underline{D}_1(a))$ is greater (smaller, resp) than $u_0(\underline{D}_0)$.

The above argument is summarized in the next lemma.

Lemma 1. *Consider a sequential equilibrium strategy profile σ^* under mechanism (M, C, D).*[7]

(i) $M_1^* \subset \cup_{a \in \mathcal{A}} M_1(a)$, $M_0^* \subset M_0$.

(ii) The outsider's equilibrium strategy $\sigma_{\mathcal{A}}^*$ maximizes (4), given $P^*(a)$.

(iii) a) Furthermore, if M_0^* and M_1^* are nonempty (i.e., σ^* is a separating equilibrium), the agent's messaging strategy σ_M^* satisfies the following.

Case 1. If $u_1(a, \underline{D}_1(a)) > u_0(\underline{D}_0)$, then $\left[\sigma_M^*(m|a) > 0 \Rightarrow \left\{m \in M_1^* \text{ and } D(m) = \underline{D}_1(a)\right\}\right]$ and thus $P^*(a) = 1$.

Case 2. If $u_1(a, \underline{D}_1(a)) > u_0(\underline{D}_0)$, then $\left[\sigma_M^*(m|a) > 0 \Rightarrow \left\{m \in M_0^* \text{ and } D(m) = \underline{D}_0\right\}\right]$ and thus $P^*(a) = 0$.

Case 3. Otherwise, then $\left[\sigma_M^*(m|a) > 0 \Rightarrow \left\{m \in M_1^* \cup M_0^* \text{ and } D(m) = \underline{D}_1(a) = \underline{D}_0\right\}\right]$ and thus $P^*(a) \in [0, 1]$

b) If $M_0^* = \emptyset$ (a pooling-continuation equilibrium), $M_1^* \neq \emptyset$ and $P^*(a) = 1$ for all $a \in \mathcal{A}$.

c) If $M_1^* = \emptyset$ (a pooling-termination equilibrium), $M_0^* \neq \emptyset$ and $P^*(a) = 0$ for all $a \in \mathcal{A}$.

4 Revelation principle

From an analogy with the standard revelation principle in a Bayesian game, we expect an arbitrary mechanism to reduce to a direct mechanism in which the agent reports directly his

[7] So far we do not rely on consistency of the belief to characterize the optimal strategies. These properties hold without it, i.e., in any (weak) perfect Bayesian equilibria, not only in sequential equilibria.

private information, namely, the outsider's action a. But if the project terminates, a is no longer related with the agent's payoff. There may be no incentive for the agent to tell the outside's action. Hence, we may not expect the agent to report a if he foresees termination of the project.

Therefore, we need to modify the direct mechanism. First, the message space should be the union of a set of the outsider's actions $A_1 \subset \mathcal{A}$ (possibly only a proper subset of $\mathcal{A}$ or the entire set $\mathcal{A}$) and the set M_0 of messages that *may* not directly tell the outsider's action; either A_1 or M_0 can be an empty set. The interim continuation schedule should let the project continue if the agent reports any of messages in A_1 and let it terminate if he announces any in M_0. We call such a mechanism a quasi-direct mechanism.

Definition 2 (quasi-direct mechanism)**.** Mechanism (M, C, D) is a **quasi-direct mechanism** if

1) the message space M is decomposed as $M = \mathcal{A}_1 \cup M_0$ with $\mathcal{A}_1 \subset \mathcal{A}$ and $\mathcal{A}_1 \cap M_0 = \emptyset$; and,

2) the principal approves continuation if the agent reports any $\tilde{a} \in \mathcal{A}_1$, and rejects continuation if he announces any $m_0 \in M_0$: i.e., $C(\tilde{a}) = 1$ for any $\tilde{a} \in A_1$ and $C(m_0) = 0$ for any $m_0 \in M_0$.

We can convert any mechanism with an arbitrary message space M to a quasi-direct mechanism with $\hat{M} = A_1^* \cup M_0^*$, while preserving the equilibrium outcome. This is our revelation principle, which is summarized as follows. See the appendix for the formal statement and proof. Note that mechanism (M, C, D) in a pooling-termination equilibrium is trivially converted to a quasi-direct mechanism just by discarding unsent messages from the message space.

Theorem 1 (Revelation principle)**.** *Consider a separating or pooling-continuation sequential equilibrium (σ^*, μ^*)—consequently, $M_1^* \neq \emptyset$—in an arbitrary mechanism (M, C, D). Let P^* be the ex-ante continuation probability, derived in* (2) *from (σ^*, μ^*). We can truthfully implement the same P^* in a sequential equilibrium $(\hat{\sigma}^*, \hat{\mu}^*)$ under a quasi-direct mechanism $(\hat{M}, \hat{C}, \hat{D})$, specified as below:*

1) The message space is $\hat{M} = A_1^* \cup M_0^*$, where $A_1^* \neq \emptyset$ and M_0^* are of the original equilibrium as in (1) *and* (3).

2) The interim continuation schedule is $\hat{C}(m_0) = 0$ for any $m_0 \in M_0^*$ and $\hat{C}(\tilde{a}) = 1$ for any $\tilde{a} \in A_1^*$.

3) The payment schedule is $\hat{D}(m_0) = \underline{D}_0$ for any $m_0 \in M_0^*$ and $\hat{D}(\tilde{a}) = \underline{D}_1(\tilde{a})$ for any $\tilde{a} \in A_1^*$.

4) The agent sends message $m_0 \in M_0^*$ with the same conditional probability as in the original equilibrium: $\hat{\sigma}_M^*(m_0|a) = \sigma_M^*(m_0|a)$ for all $a \in \mathcal{A}$ and $m_0 \in M_0$. The posterior belief conditional on $m_0 \in M_0^*$ is the same as the original mechanism: $\hat{\mu}^*(a|m_0) = \mu^*(a|m_0)$ for all $a \in \mathcal{A}$ and $m_0 \in M_0$.

5) If the outsider takes action $a \in A_1^*$, the agent announces it truthfully, i.e., sends message $a \in A_1^* \subset \hat{M}$ with probability $P^*(a)$: that is, $\hat{\sigma}_M^*(a|a) = P^*(a)$ if $a \in A_1^*$.[8] *The posterior belief after receiving*

[8]Notice $P^*(a) + \sum_{m_0 \in M_0^*} \sigma_M^*(m_0|a) = 1$ for any $a \in \mathcal{A}$ and $P^*(a) = 0$ if $a \notin A_1^*$ in the original equilibrium. The former assures that the new equilibrium satisfies $\hat{\sigma}_M^*(a|a) + \sum_{m_0 \in M_0^*} \hat{\sigma}_M^*(m_0|a) = 1$ for any $a \in \mathcal{A}_1^*$.

message $\tilde{a} \in A_1^*$ assigns probability 1 to action $\tilde{a}$: that is, $\hat{\mu}^*(\tilde{a}|\tilde{a}) = 1$ and $\hat{\mu}^*(a|\tilde{a}) = 0$ for all $a \neq \tilde{a}$.

6) *The outsider's action strategy $\hat{\sigma}_{\mathcal{A}}^*$ is the same $\sigma_{\mathcal{A}}^*$ as in the original equilibrium.*

5 Limited liability

Because an arbitrary mechanism can reduce to a quasi-direct mechanism, it is legitimate to formulate the limited liability constraint based on a quasi-direct mechanism. Denote by $L_c(a)$ the agent's liquidity holding after the outsider makes action a and continuation $c = 1$ or termination $c = 0$ of the project has been determined but before the agent pays any money to the principal.

Limited liability means that monetary payment D should be within the agent's liquidity holding at the time of payment; otherwise the monetary payment is not feasible. Generally in the presence of asymmetric information, liquidity holding is not directly observed and has to be inferred from the agent's voluntary report. In standard financial contract theory where private information is an exogenously given state ω, we usually presume truthful implementation under a direct mechanism thanks to revelation principle. Then, each message directly tells the true state and thus uniquely pins down the amount of liquidity holding in the state. The limited liability constraint just requires that, after the direct report of state ω, the promised payment should be within the anticipated liquidity holding in state ω, say $L(\omega)$. This is legitimate because the report of state ω is believed to be true and the agent is believed to really have just $L(\omega)$.

By the same token, we impose $D(a) \leq L_1(a)$ when the agent reports the outsider's action $a \in A_1^*$. What would be a legitimate form of the limited liability constraint if the agent sends message $m_0 \in M_0^*$? Message m_0 may not identify an action and several actions can still remain possible: there may be multiple actions $a \in \mathcal{A}$ s.t. $\mu^*(a|m_0) > 0$. The limited liability constraint should be formulated as requiring $D(m_0) \leq L(a)$ for any of such actions. Suppose that this constraint is not satisfied at some action a. When the agent announces message m_0, the principal believes that this action a may have been taken with some positive probability. She anticipates a possibility that liquidity holding $L(a)$ cannot cover promised payment $D(m_0)$ and thus the payment must be cut.[9] So, the principal finds that $D(m_0)$ will not be the amount of the actual payment.

In sum, we formulate limited liability constraint in a quasi-direct mechanism as

$$D(a) \leq L(a) \qquad \text{for each } a \in A_1^*; \text{ and}$$

$$D(m_0) \leq L(a) \text{ whenever } \mu^*(a|m_0) > 0 \qquad \text{for each } a \in \mathcal{A} \text{ and } m_0 \in M_0^*.$$

[9] Furthermore, because the principal cannot tell which action is taken, the agent could insist that this a has been taken and that the repayment should be cut, *even if* the true action was not a.

6 Example: financial contract and predation

Now we apply the revelation principle to the financial contract in Example 1, and find the condition for a pure-strategy sequential equilibrium in which the entrant eventually stays in the market. Our revelation principle allows us to reduce the message space to $A_1^* \cup M_0^*$, where A_1^* is the set of the incumbent's actions that let the entrant's business continue and M_0^* is the set of the messages that let the entrant quit his business.

By linearity of u_1 in d, property 3) in the revelation principle implies that the repayment should be constant for any messages that induce the same interim continuation schedule. That is, the payment schedule D is identified by two constants $\underline{D}_0$ and $\underline{D}_1$ such that

$$D(m_0) = \underline{D}_0 \quad \text{for all } m_0 \in M_0^*, \tag{5}$$

$$D(a) = \underline{D}_1 \quad \text{for all } a \in A_1^*, \tag{6}$$

Further, sequential rationality of the agent's messaging strategy implies

$$U_1(a) - \underline{D}_1 \geq -\underline{D}_0 \text{ for all } a \in A_1^*. \tag{7}$$

Any of the incumbent's actions in A_1^* should make the entrant willing to continue his business. In truthful implementation under the quasi-direct mechanism, the entrant should announce the true value of a and thus he should be willing to pay $D(a) = \underline{D}_1$. To make this announcement compatible with the entrant's incentive, his payoff from this message should not be smaller than what he could earn from other messages, especially any terminating messages in M_0^*. Condition (7) is indeed the incentive compatibility condition for the agent to reveal private information in the standard financial contract.

The last condition (7) creates threat of predation. That is, the incumbent can drive the entrant out of the market by raising the intensity of predatory conducts a so that $U_1(a)$ becomes lower than $\underline{D}_1 - \underline{D}_0$. We are interested in the contract that discourages the incumbent from predation. Let $\bar{a}^P$ be the incumbent's maximal intensity of predatory conducts that yields a higher post-monopoly profit than the maximal duopoly profit:

$$\bar{a}^P := \max \left\{ a \in \mathcal{A} \middle| V_0(a) \geq V_1(a^*) \right\},$$

where a^* is the incumbent's optimal action in the case the entrant stays in the market: i.e., $a^* := \arg\max_{a \in \mathcal{A}} V_1(a)$. Since $V_0(a) \geq V_1(a)$ for any a, we have $\bar{a}^P \geq a^*$. To prevent the entrant from being driven out, the entrant should be allowed to continue the project even after this maximal intensity of predatory conduct takes place:

$$\bar{a}^P \in A_1^*. \tag{8}$$

For feasibility of the repayment schedule, we impose the limited liability constraint:

$$\underline{D}_0 \leq 0 \quad \text{for all } m_0 \in M_0^*, \tag{9}$$

$$\underline{D}_1 \leq U_1(\tilde{a}) \quad \text{for all } \tilde{a} \in A_1^*. \tag{10}$$

Finally, to agree on the contract, the investor should not suffer a loss in the equilibrium play. That is, the repayment in the case of continuation of the business should cover the fixed costs:

$$\underline{D}_1 \geq F. \tag{11}$$

This is the participation (individual rationality) condition for the principal in the standard financial contract.

Combining the non-predation condition (8) and the limited liability constraint (10) (for the first inequality below) and then the participation condition (11) (for the second), we obtain a necessary condition for an entrant to stay in the market in the equilibrium outcome:

$$U_1(\bar{a}^P) \geq \underline{D}_1 \geq F, \qquad \therefore \qquad U_1(\bar{a}^P) \geq F.$$

The incumbent gives up predation and chooses a^* only if this condition is met. Note that $\bar{a}^P \geq a^*$ and thus $U_1(a^*) \geq U_1(\bar{a}^P)$ Therefore, even if the entrant's business could generate a positive net profit $U_1(a^*) - F$ in the equilibrium outcome without threat of predation, the entrant may not be able to finance the entry cost.

Zusai (2022) extends this model to allow the entrant to choose the scale of his own business and also to raise precautionary liquidity upon the entry and discusses the relation between threat of predation and excess demand of precautionary liqudity.

7 Concluding remarks

We prove the revelation principle for sequential equilibria in a specific class of complete but imperfect information games. In our game, a principal makes a binary choice about continuation of a project whose outcome is affected by an "outsider" of the contract. The contract should be designed to elicit information from an agent who can observe the outsider's action. But the outsider himself is excluded from the contract, for example, because the outsider is a competitor against the agent in product market competition. While we do not assume uncertainty about payoff functions, we pay attention to strategic uncertainty that the outsider's action might deviate from equilibrium; so, we adopt sequential equilibrium as a solution concept. Like other versions, our version of revelation principle allows us to focus on truthful implementation in a (quasi-)direct mechanism without restricting implementable outcomes in this setting.

In the paper, the agent is assumed to choose only a message to the principal. Our revelation principle is straightforwardly applicable to a game in which the agent also makes an action before the outsider does, as long as the agent's action is observable and verifiable. for the principal Furthermore, with the assumption of verifiablity of the agent's action and some trivial modifications of the proof, we can easily extend it to a game in which both the entrant and the outsider simultaneously make actions.[10]

[10]Zusai (2012) considers a Cournot-competition version of the model in the last section and directly proves the

In various situations, endogenous strategic uncertainty would be a central issue and have more importance for its economic outcome than exogenous physical uncertainty. Predatory pricing is one of such situations, though physical uncertainty has been added to preceding models to formulate the situation as a Bayesian game. Our version of revelation principle allows applied theorists to evaluate effects of strategic uncertainty on economic outcomes in a simple model without complicated techniques.

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revelation principle specifically to it.

A Formal presentation and proof of the revelation principle

Theorem 1. *Suppose that a mixed (behavioral) strategy profile* $\sigma^* = \{\sigma_1^*, \sigma_{\mathcal{A}}^*, \sigma_M^*\}$ *is a sequential equilibrium under mechanism* (M, C, D) *with message space* M*, interim continuation schedule* $c : M \to \{0, 1\}$ *and payment schedule* $D : M \to \mathbb{R}$*.*

If $M_1^* \neq \emptyset$*, there exists a sequential equilibrium* $(\hat{\sigma}^*, \hat{\mu}^*)$ *that results in the same strategy of the outsider* $\sigma_{\mathcal{A}}^*$ *and the same continuation probability* $P^* : \mathcal{A} \to [0, 1]$ *under a quasi-direct mechanism* $(\hat{M}, \hat{C}, \hat{D})$ *with message space* $\hat{M} = M_0^* \cup A_1^*$*, interim continuation schedule* $\hat{C} : \hat{M} \to \{0, 1\}$ *and payment schedule* $\hat{D} : \hat{M} \to \mathbb{R}$ *such as*

$$\begin{cases} \hat{C}(\tilde{a}) = 1, \quad \hat{D}(\tilde{a}) := \underline{D}_1(\tilde{a}) & \text{for each } \tilde{a} \in A_1^*, \\ \hat{C}(m) = 0, \quad \hat{D}(m) := \underline{D}_0 & \text{for each } m \in M_0^*. \end{cases} \tag{12}$$

The pair $(\hat{\sigma}^*, \hat{\mu}^*)$ *is specified as follows:*

$$(\hat{\sigma}_{\mathcal{A}}^*) \; \hat{\sigma}_{\mathcal{A}}^*(a) := \sigma_{\mathcal{A}}^*(a) \quad \text{for each } a \in \mathcal{A};$$

$$(\hat{\sigma}_M^*) \begin{cases} \hat{\sigma}_M^*(\tilde{a}|a) := 0 & \text{for each } a \in \mathcal{A}, \tilde{a} \in A_1^* \setminus \{a\}, \\ \hat{\sigma}_M^*(a|a) := P^*(a) & \text{for each } a \in A_1^*, \\ \hat{\sigma}_M^*(m|a) := \sigma_M^*(m|a) & \text{for each } a \in \mathcal{A}, m \in M_0^*, \end{cases}$$

$$(\hat{\mu}^*) \begin{cases} \hat{\mu}^*(a|\tilde{a}) := I(a, \tilde{a}) & \text{for each } a \in \mathcal{A}, \tilde{a} \in A_1^*, \\ \hat{\mu}^*(a|m) := \mu^*(a|m) & \text{for each } a \in \mathcal{A}, m \in M_0^*. \end{cases}$$

Here $I(a, \tilde{a})$ *is the indicator function for* $a = \tilde{a}$*: i.e.,* $I(a, \tilde{a})$ *is* 1 *if* $a = \tilde{a}$ *and* 0 *otherwise.*

Proof. We show that the strategy profile $\hat{\sigma}^* = \{\sigma_{\mathcal{A}}^*, \hat{\sigma}_M^*\}$ specified in the theorem is a sequential equilibrium under the belief $\hat{\mu}^*$.

Consistency of belief.

From the sequence of completely mixed strategy profiles $\{\sigma^k\}$ converging to σ^* in the original sequential equilibrium, we define the sequence $\{\hat{\sigma}^k\}$ and then prove the consistency of the belief $\hat{\mu}^*$.

For each $k \in \mathbb{N}$, define $\phi^k : M_0 \to (0, \#\mathcal{A}]$ and $\Phi^k : M_0 \to (0, 1)$ as

$$\phi^k(m) := \sum_{a \in \mathcal{A}} \sigma_M^k(m|a), \quad \Phi^k(m) := \sum_{a \in \mathcal{A}} \sigma_{\mathcal{A}}^k(a) \sigma_M^k(m|a),$$

which converge to $\phi^*(m) := \sum_{a \in \mathcal{A}} \sigma_M^*(m|a) \in [0, \#\mathcal{A}]$ and $\Phi^*(m) := \sum_{a \in \mathcal{A}} \sigma_{\mathcal{A}}^*(a) \sigma_M^*(m|a) \in [0, 1]$, respectively, as $k \to \infty$. Let $\Phi_0^k \in (0, 1)$ be $\Phi_0^k := \min\{\Phi^k(m) | m \in M_0^*\}$, which converges to $\Phi_0^* := \min\{\Phi^*(m) | m \in M_0^*\} \in [0, 1]$.

For each $k \in \mathbb{N}$, define $\hat{\sigma}_{\mathcal{A}}^k \in (0, 1)$ as

$$\hat{\sigma}_{\mathcal{A}}^k(a) := \frac{\Phi_0^k}{\sqrt{k} \#\mathcal{A}} + \left(1 - \frac{\Phi_0^k}{\sqrt{k}}\right) \sigma_{\mathcal{A}}^k(a).$$

Since $\sigma^k_{\mathcal{A}} \to \sigma^*_{\mathcal{A}}$ and $\Phi^k_0 \to \Phi^*_0 \in [0,1]$, we have $\hat{\sigma}^k_{\mathcal{A}} \to \sigma^*_{\mathcal{A}} = \hat{\sigma}^*_{\mathcal{A}}$ as $k \to \infty$.

For each $k \in \mathbb{N}$, define $\hat{\sigma}^k_M$ as

$$\begin{aligned}
\hat{\sigma}^k_M(\tilde{a}|a) &:= \frac{\Phi^k_0}{k\#A^*_1} + \left(1 - \frac{\Phi^k_0}{k}\right) I(a,\tilde{a})P^k(a) && \in (0,1), \\
\hat{\sigma}^k_M(m|a) &:= \left(1 - \frac{\Phi^k_0}{k}\right) \sigma^k_M(m|a) \frac{1 - I^*_1(a)P^k(a)}{1 - P^k(a)} && \in (0,1)
\end{aligned}$$

for each $a \in \mathcal{A}, \tilde{a} \in A^*_1, m \in M^*_0$. Here $P^k : \mathcal{A} \to (0,1)$ is given by

$$P^k(a) := \sum_{m' \in M/M^*_0} \sigma^k_M(m'|a) \ \in (0,1),$$

and $I^*_1(a)$ is the indicator for $a \in A^*_1$: i.e., $I^*_1(a)$ is 1 if $a \in A^*_1$ and 0 otherwise. $\hat{\sigma}^k_M(\cdot|a)$ belongs to the interior of $\Delta\hat{M}$ for all $a \in \mathcal{A}$, since $\hat{\sigma}^k_M(\tilde{a}|a), \hat{\sigma}^k_M(m|a) \in (0,1)$ and

$$\begin{aligned}
&\sum_{\tilde{a} \in A^*_1} \hat{\sigma}^k_M(\tilde{a}|a) + \sum_{m' \in M_0} \hat{\sigma}^k_M(m'|a) \\
=&\frac{\Phi^k_0}{k} + \left(1 - \frac{\Phi^k_0}{k}\right) \left\{ \sum_{\tilde{a} \in A^*_1} I(a,\tilde{a})P^k(a) + \sum_{m \in M^*_0} \sigma^k_M(m|a) \frac{1 - I^*_1(a)P^k(a)}{1 - P^k(a)} \right\} \\
=&\frac{\Phi^k_0}{k} + \left(1 - \frac{\Phi^k_0}{k}\right) \left\{ I^*_1(a)P^k(a) + 1 - I^*_1(a)P^k(a) \right\} = 1.
\end{aligned}$$

Here we use the identities $\sum_{\tilde{a} \in A^*_1} I(a,\tilde{a}) = I^*_1(a)$ and $P^k(a) + \sum_{m \in M^*_0} \sigma^k_M(m|a) = 1$. According to Lemma 1 (iii), we obtain $P^k \to P^*$ and thus $\hat{\sigma}^k_M \to \hat{\sigma}^*_M$ as $k \to \infty$.

The Bayesian belief $\hat{\mu}^k$, determined from $(\hat{\sigma}^k_{\mathcal{A}}, \hat{\sigma}^k_M)$, actually converges to $\hat{\mu}^*$. For each $a \in \mathcal{A}, m \in M_0$, the belief is

$$\begin{aligned}
&\hat{\mu}^k(a|m) \\
&:= \frac{\hat{\sigma}^k_M(m|a)\hat{\sigma}^k_{\mathcal{A}}(a)}{\sum_{a \in \mathcal{A}} \hat{\sigma}^k_M(m|a)\hat{\sigma}^k_{\mathcal{A}}(a)} \\
&= \frac{\left(1 - \frac{\Phi^k_0}{k}\right) \sigma^k_M(m|a) \frac{1 - I^*_1(a)P^k(a)}{1 - P^k(a)} \left\{ \frac{\Phi^k_0}{\sqrt{k}\#\mathcal{A}} + \left(1 - \frac{\Phi^k_0}{\sqrt{k}}\right) \sigma^k_{\mathcal{A}}(a) \right\}}{\sum_{a' \in \mathcal{A}} \left(1 - \frac{\Phi^k_0}{k}\right) \sigma^k_M(m|a') \frac{1 - I^*_1(a')P^k(a')}{1 - P^k(a')} \left\{ \frac{\Phi^k_0}{\sqrt{k}\#\mathcal{A}} + \left(1 - \frac{\Phi^k_0}{\sqrt{k}}\right) \sigma^k_{\mathcal{A}}(a') \right\}} = \frac{N^k(a)}{\bar{N}^k}.
\end{aligned}$$

Here $N^k(a)$ and $\bar{N}^k$ are defined as

$$\begin{aligned}
N^k(a) &:= \left[\sigma^k_M(m|a) \frac{1 - I^*_1(a)P^k(a)}{1 - P^k(a)} \left\{ \frac{\Phi^k_0}{\sqrt{k}\#\mathcal{A}} + \left(1 - \frac{\Phi^k_0}{\sqrt{k}}\right) \sigma^k_{\mathcal{A}}(a) \right\} \right] \Big/ \left[\left(1 - \frac{\Phi^k_0}{\sqrt{k}}\right) \Phi^k(m) \right] \\
&= \left\{ (\sqrt{k}\#\mathcal{A})^{-1} \left(1 - \frac{\Phi^k_0}{\sqrt{k}}\right)^{-1} \frac{\Phi^k_0}{\Phi^k(m)} \sigma^k_M(m|a) + \mu^k(a|m) \right\} \frac{1 - I^*_1(a)P^k(a)}{1 - P^k(a)}, \\
\bar{N}^k &:= \sum_{a' \in \mathcal{A}} N^k(a') = \sum_{a' \in \mathcal{A}} \left\{ (\sqrt{k}\#\mathcal{A})^{-1} \left(1 - \frac{\Phi^k_0}{\sqrt{k}}\right)^{-1} \frac{\Phi^k_0}{\Phi^k(m)} \sigma^k_M(m|a') + \mu^k(a'|m) \right\} \frac{1 - I^*_1(a')P^k(a')}{1 - P^k(a')},
\end{aligned}$$

$$= (\sqrt{k}\#\mathcal{A})^{-1}\left(1-\frac{\Phi_0^k}{\sqrt{k}}\right)^{-1}\frac{\Phi_0^k}{\Phi^k(m)}\left\{\phi^k(m)+\sum_{a'\notin A_1^*}\sigma_M^k(m|a')\frac{P^k(a')}{1-P^k(a')}\right\}$$
$$+1+\sum_{a'\notin A_1^*}\mu^k(a'|m)\frac{P^k(a')}{1-P^k(a')}. \tag{13}$$

First, calculate the limit of N^k. If $a \in A_1^*$, the fraction $(1 - I_1^*(a)P^k(a))/(1 - P^k(a))$ is equal to 1 for all $k \in \mathbb{N}$ and thus converges to 1. Otherwise, the fraction is $1/(1 - P^k(a))$ and converges to $1/1 = 1$. Since $P^k(a) \to P^*(a) = 0$, the fraction converges to 1 in both cases. Because $\Phi_0^k \leq \Phi^k(m)$ by definition, we have

$$0 \leq (\sqrt{k}\#\mathcal{A})^{-1}\left(1-\frac{\Phi_0^k}{k}\right)^{-1}\frac{\Phi_0^k}{\Phi^k(m)}\sigma_M^k(m|a) \leq (\sqrt{k}\#\mathcal{A})^{-1}\left(1-\frac{\Phi_0^k}{k}\right)^{-1}\sigma_M^k(m|a).$$

The RHS converges to $0 \cdot 1 \cdot \sigma_M^*(m|a) = 0$ and so does the middle term. Therefore, N^k converges to $\{0 + \mu^*(a|m)\} \cdot 1 = \mu^*(a|m)$.

Next, calculate the limit of D^k. Let $\bar{r}^k := \max\{P^k(a)/(1 - P^k(a))|a \notin A_1^*\}$. This converges to 0 because $P^k(a)/(1-P^k(a)) \to P^*(a)/(1-P^*(a)) = 0/1 = 0$ for any $a \notin A_1^*$. Because $\Phi_0^k \leq \Phi^k(m)$ and $\sum_{a'\notin A_1^*}\sigma_M^k(m|a') \leq \phi^k(m)$, the first term in (13) is *at most* $(\sqrt{k}\#\mathcal{A})^{-1}\left(1-\frac{\Phi_0^k}{\sqrt{k}}\right)^{-1}(1+\bar{r}^k)\phi^k(m)$, which converges to $0 \cdot 1 \cdot 1 \cdot \phi^*(m) = 0$; as this term is at least 0 for all $k \in \mathbb{N}$, it also converges to 0. Likewise, because $\sum_{a'\notin A_1^*}\mu^k(a'|m) \leq \Phi^k(m)$, the third term in (13) is *at most* $\bar{r}^k$, which converges to 0, and thus this term also converges to 0 as it is at least 0 for all $k \in \mathbb{N}$. Hence D^k converges to $0 + 1 + 0 = 1$. Therefore, we have

$$\lim_{k\to\infty}\hat{\mu}^k(a|m) = \lim_{k\to\infty}N^k/D^k = \mu^*(a|m)/1 = \mu^*(a|m) = \hat{\mu}^*(a|m).$$

For $\tilde{a} \in A_1^*, a \in \mathcal{A}/\{\tilde{a}\}$, the belief is

$$\hat{\mu}^k(a|\tilde{a}) := \frac{\hat{\sigma}_M^k(\tilde{a}|a)\hat{\sigma}_\mathcal{A}^k(a)}{\sum_{a\in\mathcal{A}}\hat{\sigma}_M^k(\tilde{a}|a)\hat{\sigma}_\mathcal{A}^k(a)} = \frac{\frac{\Phi_0^k}{k\#A_1^*}\hat{\sigma}_\mathcal{A}^k(a)}{\frac{\Phi_0^k}{k\#A_1^*}+\left(1-\frac{\Phi_0^k}{k}\right)P^k(\tilde{a})\hat{\sigma}_\mathcal{A}^k(\tilde{a})}$$
$$= \left[\frac{1}{\hat{\sigma}_\mathcal{A}^k(a)}+\left(\frac{k}{\Phi_0^k}-1\right)\#A_1^*\frac{\hat{\sigma}_\mathcal{A}^k(\tilde{a})}{\hat{\sigma}_\mathcal{A}^k(a)}P^k(\tilde{a})\right]^{-1}$$
$$< \left[\left(\frac{k}{\Phi_0^k}-1\right)\#A_1^*\frac{\hat{\sigma}_\mathcal{A}^k(\tilde{a})}{\hat{\sigma}_\mathcal{A}^k(a)}P^k(\tilde{a})\right]^{-1} = \left[\left(\frac{k}{\Phi_0^k}-1\right)\frac{\Phi_0^k+\left(\sqrt{k}-\Phi_0^k\right)\#\mathcal{A}\sigma_\mathcal{A}^k(\tilde{a})}{\Phi_0^k+\left(\sqrt{k}-\Phi_0^k\right)\#\mathcal{A}\sigma_\mathcal{A}^k(a)}P^k(\tilde{a})\right]^{-1}$$
$$< \left[\left(\frac{k}{\Phi_0^k}-1\right)\#A_1^*\frac{\Phi_0^k}{\Phi_0^k+\left(\sqrt{k}-\Phi_0^k\right)\#\mathcal{A}}P^k(\tilde{a})\right]^{-1}$$
$$= \left(\frac{\Phi_0^k}{k-\Phi_0^k}\cdot\frac{1}{\#A_1^*}+\frac{\sqrt{k}-\Phi_0^k}{k-\Phi_0^k}\cdot\frac{\#\mathcal{A}}{\#A_1^*}\right)\frac{1}{P^k(\tilde{a})}.$$

The first strict inequality comes from $\hat{\sigma}_\mathcal{A}^k(a) > 0$ and the second is from $\sigma_\mathcal{A}^k(\tilde{a}) > 0, \sigma_\mathcal{A}^k(a) < 1$

and $\sqrt{k} \geq 1 > \Phi_0^k$. As $\hat{\mu}^k(a|\tilde{a}) > 0$ and $P^*(\tilde{a}) > 0$ for any $\tilde{a} \in A_1^*$, this implies

$$0 \leq \lim_{k\to\infty} \hat{\mu}^k(a|\tilde{a}) \leq 0 \cdot \frac{1}{P^*(\tilde{a})} = 0,$$
$$\therefore \quad \lim_{k\to\infty} \hat{\mu}^k(a|\tilde{a}) = 0 = \hat{\mu}^*(a|\tilde{a}).$$

Because this holds for all $a \in \mathcal{A}/\{\tilde{a}\}$, we have

$$\lim_{k\to\infty} \hat{\mu}^k(\tilde{a}|\tilde{a}) = 1 = \hat{\mu}^*(\tilde{a}|\tilde{a}).$$

Therefore the belief $\hat{\mu}^*$ specified in ($\hat{\mu}^*$) is actually consistent with $\hat{\sigma}^*$.

Sequential rationality.

We prove the optimality of strategy profile $\hat{\sigma}^*$ given belief $\hat{\mu}^*$. First, we check the optimality of the message strategy. Consider the optimal messaging strategy after $a \in A_1^*$. By definition, it implies $P^*(a) > 0$; furthermore, by applying Lemma 1 (iii) to the original equilibrium, we find $u_1(a, \underline{D}_1(a)) \geq u_0(\underline{D}_0)$. So, report of a is at least as good for the entrant as any other report. In particular, if $P^*(a) \in (0,1)$, Lemma 1 (iii) suggests that $u_1(a, \underline{D}_1(a)) = u_0(\underline{D}_0)$ and thus the agent is indifference between report of a and report of any $m \in M_0^*$. On contrary, if $a \notin A_1^*$, we have $P^*(a) = 0$ by definition and $u_1(a, \underline{D}_1(a)) \leq u_0(\underline{D}_0)$ by Lemma 1 (iii); report of any $m \in M_0^*$ is at least as good for the entrant as any other report. Therefore, $\hat{\sigma}_M^*$ specified in ($\hat{\sigma}^*$) is an optimal strategy.

Given $\hat{\sigma}_M^*$, the continuation probability after each a is the same as $P^*(a)$ in the original equilibrium. Hence the outsider's expected profit after action a remains the same. Therefore, $\hat{\sigma}_{\mathcal{A}}^* \equiv \sigma_{\mathcal{A}}^*$ is still the optimal strategy in equilibrium $(\hat{\sigma}^*, \hat{\mu}^*)$.

We have established the consistency of belief $\hat{\mu}^*$ with $\hat{\sigma}^*$ and the sequential rationality of the strategy profile $\hat{\sigma}^*$, and thus the profile $(\hat{\mu}^*, \hat{\sigma}^*)$ is a sequential equilibrium under quasi-direct mechanism $(\hat{M}, \hat{C}, \hat{D})$. □