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Adjustment costs in dynamically optimal pricing of a network good

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Abstract

In this note, we consider dynamically optimal pricing of a network good, when consumers' demand adjusts only gradually. We find a Lyapunov function that characterizes where and how the platform size converges under the dynamically optimal pricing. Given the current platform size, we compare the value of the Lyapunov function with the profit under static pricing that keeps this current size at a Nash equilibrium of consumers' entry game. We show that the difference between them can be interpreted as adjustment costs and we justify this interpretation by regarding a myopic pricing scheme as an approximation. This justification suggests that recurrent adjustment of the myopic pricing scheme brings the platform to the same size in the long run as the dynamically optimal pricing.
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1 Introduction

It is common for a dynamic optimization problem to have a stationary state different from static optimum. Solow's modified golden rule in macroeconomic growth theory is a classic example. In that theory, the difference is explained from intertemporal discounting of future consumption. Different models need different interpretations suitable to their contexts.

Dhebar and Oren (1985) is a seminal paper on dynamically optimal pricing of a network good. Here we revisit their model.[1] An owner of a proprietary platform dynamically adjusts the fee level to maximize the discounted total profit. The dynamic nature comes from the assumption that the platform size changes only gradually toward a temporary equilibrium given the current fee level. We find a Lyapunov function for the dynamic of the platform size: its maximum coincides with the platform size at the stationary point under the dynamically optimal pricing and the function's value grows over time. The function can be interpreted as the discounted total profit in the case that the platform size immediately shifts to a Nash equilibrium, plus an additional term.

The additional term explains the difference between the static optimum and the stationary point under the dynamically optimal pricing. To interpret this term, we clarify the source of gradual adjustment of the platform size. We construct the platform dynamic from the aggregate best response dynamic of heterogeneous agents, using the result in Ely and Sandholm (2005). Then, we can decompose the fee into the gross revenue from exploitation of the utility of participation and the cost to attract target agents. In the BRD, agents change their decisions only occasionally and thus the actual size moves to the target size only gradually. There is a time lag between when the owner changes the fee to target a certain mass of agents and when they actually enter the platform and yield the revenues to the owner. Such an inertial entry process causes a loss of the exploitation revenues, which is seen as "adjustment costs."

To formalize this idea of adjustment costs and connect it mathematically with the additional term in the Lyapunov function, we consider a simplified fee scheme in which the platform owner fixes the target platform size forever while adjusting the fee level to exploit participants' utilities that grow with the platform size. Fixing the target size, it becomes clearer how much of the exploitation revenue is lost by inertia in the entry process. Furthermore, the derivative of the owner's discounted total revenue in this fee scheme with respect to the target size is proportional to the derivative of the Lyapunov function. In particular, the owner keeps the target size at the initial platform size in this fee scheme if and only if this initial size coincides with the maximum of the Lyapunov function, i.e., the size in the stationary state under the dynamically optimal pricing. This suggests that, even if the platform owner is myopic about future change in the target size, recurrent adjustment of the target size brings the platform size to the same stationary size as the dynamically optimal pricing by a fully rational platform owner.

The paper proceeds as follows. We set up the model and the dynamic optimization problem in the next section. The dynamically optimal pricing is characterized and the Lyapunov function is given in Section 3. In Section 4, we present interpretation of adjustment costs through

[1] The recent variation of the model includes Mesak et al. (2011), and Fruchter and Sigué (2013), for example.

the fixed target fee scheme and a theorem to connect it with the Lyapunov function and the dynamically optimal pricing. The proof of the theorem is in Section 5.

2 The Model

In Dhebar and Oren (1985), the platform owner maximizes the discounted total revenue:

$$\max_{\{(P_t,n_t)\}_{t\in\mathbb{R}_+}} \int_0^\infty e^{-\rho t} P_t n_t dt \quad \text{s.t. } \dot{n}_t = W(P_t, n_t),$$

where $\rho > 0$ is the discount rate, P_t is the fee charged to each participant of the platform at time $t \in \mathbb{R}_+$ and n_t is the number of participants. We assume a unit mass of infinitely many agents (potential participants) and thus allow n_t to take any real number in $[0, 1]$.

It is useful for interpretation of the fee scheme to reformulate the platform dynamic from aggregation of individual agents' decision making. We suppose that individual decision follows the best response dynamic (Gilboa and Matsui, 1991; Hofbauer, 1995). At each moment of time $t \in \mathbb{R}_+$, an agent gets utility $u(n_t) - P_t$ if he participates in the platform and utility θ if not. Gross utility of participation $u : [0, 1] \to \mathbb{R}$ is an increasing function, which reflects positive network externality. We also assume differentiability and strict concavity of u. Utility of no participation $\theta \in [0, \bar{\theta}]$ varies among agents, and thus we call it an agent's *type*. Let F be the cumulative distribution function of agents' types with density f: $F(\theta)$ is the mass of agents whose types are not higher than θ. We assume that f is continuous and the support of F is $[0, \bar{\theta}]$.

In the best response dynamic, each agent only occasionally revises whether or not to participate: a revision opportunity follows a Poisson process with arrival rate 1 and, if an agent gets an opportunity at time t, he chooses the best action based on the current state (P_t, n_t). So the individual decision process exhibits inertia and it is based on myopic optimization.[2] According to Ely and Sandholm (2005), the aggregate best response dynamic reduces to

$$\dot{n}_t = F(u(n_t) - P_t) - n_t.$$

Note that an agent whose type is lower (higher, resp.) than $u(n_t) - P_t$ should choose to participate (not to participate, resp.) if he gets an opportunity to revise his decision. So, we can say that $F(u(n_t) - P_t)$ is the platform size in a temporary equilibrium in a static entry game given p_t, as we will clarify soon. In the aggregate BRD, the platform size gradually moves toward the temporary equilibrium at each moment of time.

We call the type equal to $u(n_t) - P_t$ the **target type** and $F(u(n_t) - P_t)$ the **target platform size** at time t. Setting the target type at $\hat{\theta}_t$ means the fee being set at $P_t = u(n_t) - \hat{\theta}_t$. We can rewrite the optimization problem in terms of target type $\hat{\theta}_t$ and current platform size n_t:

$$\max_{\{(\hat{\theta}_t,n_t)\}_{t\in\mathbb{R}_+}} \int_0^\infty e^{-\rho t} (u(n_t) - \hat{\theta}_t) n_t dt \quad \text{s.t. } \dot{n}_t = F(\hat{\theta}_t) - n_t. \tag{1}$$

[2] Dhebar and Oren (1985) consider such a myopic entry process less formally and introduce the concept of the target customers ("marginal customers" in their words). What we add here is the quasi-linear form, which allows us to decompose the fee to the exploitation revenue and the fee deduction as in Section 4.

For comparison, consider a static entry game and the corresponding static optimum. In the static game, after the owner sets the fee, each agent simultaneously decides on whether or not to join the platform. In a Nash equilibrium given fee p, platform size n satisfies $F(u(n) - p) = n$: every agent whose type is lower than the target type participates in the platform and a higher type does not.[3] In other words, the owner has to set the fee at $p = u(n) - F^{-1}(n)$ to implement platform size n in a Nash equilibrium. So, if agents immediately play such a Nash equilibrium, the owner should implement platform size n^* that maximizes her equilibrium revenue:

$$\widehat{\Pi}(n) := \{u(n) - F^{-1}(n)\}n.$$

We assume that $\widehat{\Pi}$ is concave and has an interior maximum $n^* \in (0, 1)$. The statically optimal platform size n^* satisfies

$$V'(n^*) - F^{-1}(n^*) - \frac{n^*}{f \circ F^{-1}(n^*)} = 0,$$

where $V(n) := u(n)n$ is the aggregate gross utility of participants. The statically optimal fee level is $p^* := u(n^*) - F^{-1}(n^*)$.

3 The dynamically optimal pricing

It is easy to verify that the Hamiltonian of (1) is strictly concave in target type $\hat{\theta}$ if F is concave, i.e., $f' < 0$.[4] The solution of (1) is thus characterized by the following first-order conditions with costate variable $\mu \in \mathbb{R}$:

$$\hat{\theta} = \begin{cases} \bar{\theta} & \text{if } \mu > 0, n < \mu f(\bar{\theta}), \\ f^{-1}(n/\mu) & \text{if } \mu > 0, n \geq \mu f(\bar{\theta}), \\ 0 & \text{if } \mu \leq 0, \end{cases} \tag{2a}$$

$$\dot{n} = F(\hat{\theta}) - n, \tag{2b}$$

$$\dot{\mu} = (1+\rho)\mu + \hat{\theta} - V'(n). \tag{2c}$$

We impose the transversality condition to select the solution path:

$$\lim_{t \to \infty} e^{-\rho t} \mu_t n_t = 0. \tag{3}$$

The saddle path is the only solution path that satisfies all these conditions. (See Figure 1.)

Let $(n^\$, \mu^\$)$ be the stationary point such that $\dot{n} = \dot{\mu} = 0$. Assume that $n^\$ \in (0, 1)$. The platform size $n^\$$ satisfies

$$V'(n^\$) - F^{-1}(n^\$) - (1+\rho)\frac{n^\$}{f \circ F^{-1}(n^\$)} = 0,$$

[3] As is typical of such an entry game with positive network externality, there are multiple equilibria: $n = 0$ is always an equilibrium. It is one of theoretical reasons to introduce a dynamic into the entry game.

[4] Zusai (2015) solves the dynamically optimal pricing problem for a convex F.

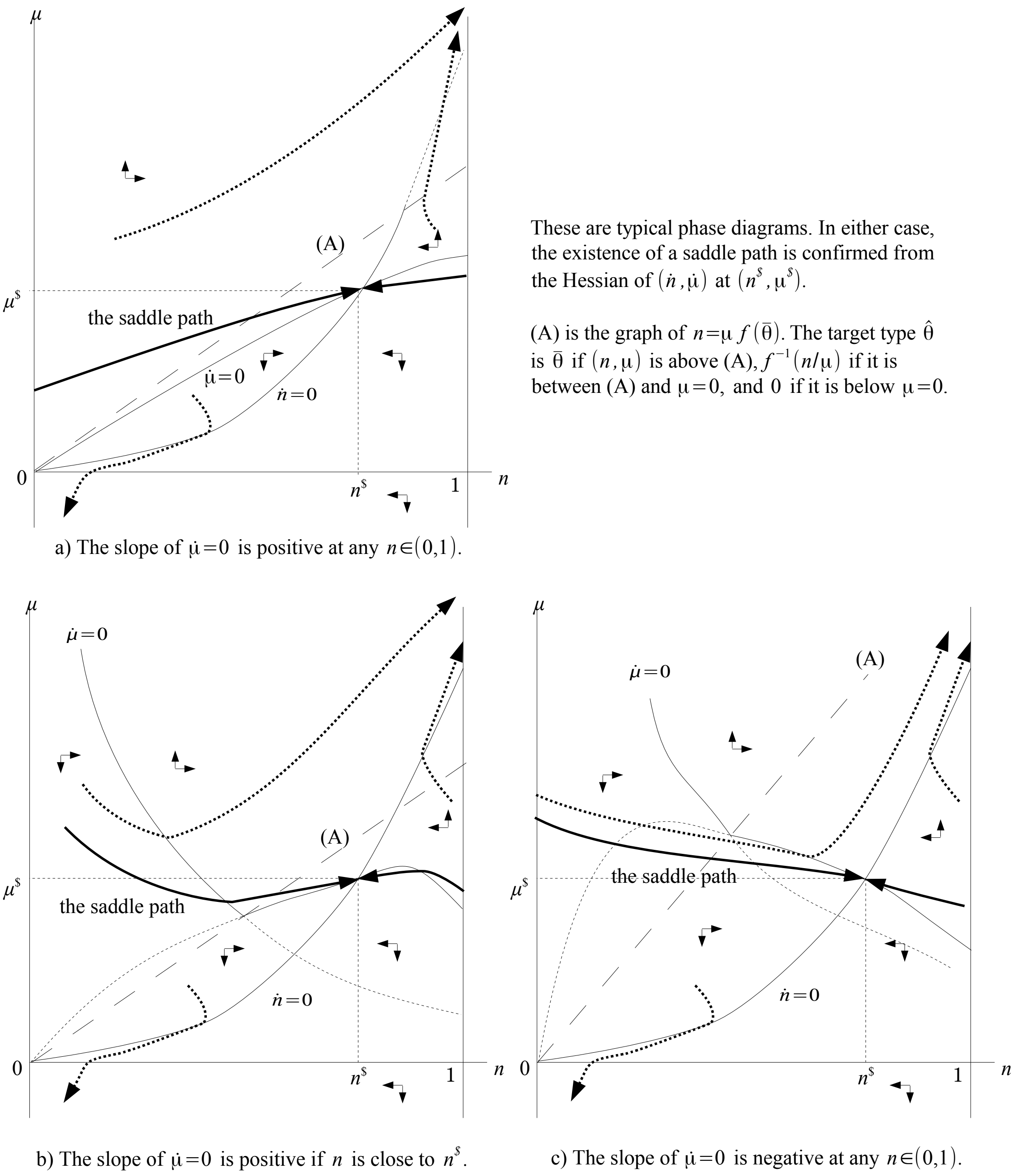

Figure 1: Phase diagram of (n, μ) under the dynamically optimal pricing. Dotted arrows are paths going to $n = 0$ or $n = 1$ and violating (3).

with $\mu^\$:= n^\$ / f \circ F^{-1}(n^\$)$. It is the maximum of function $\widetilde{\Pi}_\rho : [0,1] \to \mathbb{R}$ given by

$$\widetilde{\Pi}_\rho(n) := \frac{1}{\rho}\widehat{\Pi}(n) - \left(nF^{-1}(n) - \int_0^n F^{-1}(\nu)d\nu \right).$$

Note that $f' < 0$ and concavity of $\widehat{\Pi}$ imply strict concavity of $\widetilde{\Pi}_\rho$, which further implies the existence of a saddle path. Along the saddle path, n_t monotonically converges to $n^\$$ and thus $\widetilde{\Pi}_\rho$ increases to its maximum. So $\widetilde{\Pi}_\rho$ is a Lyapunov function of the dynamic of the platform size under the dynamically optimal pricing.

The Lyapunov function $\widetilde{\Pi}_\rho$ shows difference between the stationary point and the static optimum, which maximizes $\widehat{\Pi}$ and equivalently $\widehat{\Pi}/\rho$—the discounted total revenue in the case that the platform size immediately shifts to Nash equilibrium. It follows that the stationary size $n^\$$ is smaller than the statically optimal size n^*. Even if the initial platform size coincides with the static optimum n^*, the dynamically optimal pricing shrinks it to $n^\$ < n^*$. This pattern is common with a classical growth model, in which the capital stock in Solow's golden rule shrinks to the modified golden-rule level. In the growth model, the difference is explained from discounting of future consumption. As the discounting itself has been considered in maximization of $\widehat{\Pi}/\rho$, there remains a difference between $\widetilde{\Pi}_\rho$ and $\widehat{\Pi}/\rho$. One might attempt to explain the difference in our problem by arguing adjustment costs. We elaborate such an interpretation in the next section.

4 Adjustment Costs

We attempt to interpret the difference between $\widehat{\Pi}/\rho$ and $\widetilde{\Pi}_\rho$ as "adjustment costs" caused by the inertial entry process. For clearer formulation of this idea, consider a much simplified fee scheme in which the target type is fixed at $\hat{\theta}_\odot$ since time 0. The fee level should be adjusted accordingly at each moment of time as

$$P_t = u(n_t) - \hat{\theta}_\odot \text{ for each } t \in [0, \infty).$$

We call such a fee scheme the **fixed target fee scheme**. Note that the fee consists of two parts: *the gross revenue from exploitation of a participant's gross utility* $u(n_t)$, and *the fee deduction to attract target type* $\hat{\theta}_\odot$. The platform size grows as

$$n_t = \Psi_t(n_\odot; n_0) := e^{-t}n_0 + (1 - e^{-t})n_\odot \text{ for each } t \in [0, \infty). \tag{4}$$

We verify the relationship between the discounted total profit under this fee scheme and the Lyapunov function $\widetilde{\Pi}_\rho$.

Theorem 1. *Suppose that the platform owner keeps the target type at $\hat{\theta}_\odot$ since time 0. Denote by $\check{\Pi}(n_\odot; n_0)$ the discounted sum of the owner's profits in the whole period $(0, \infty)$.*

$$\check{\Pi}_\rho(n_\odot; n_0) := \int_0^\infty e^{-\rho t} \left\{ V(\Psi_t(n_\odot; n_0)) - F^{-1}(n_\odot)\Psi_t(n_\odot; n_0) \right\} dt.$$

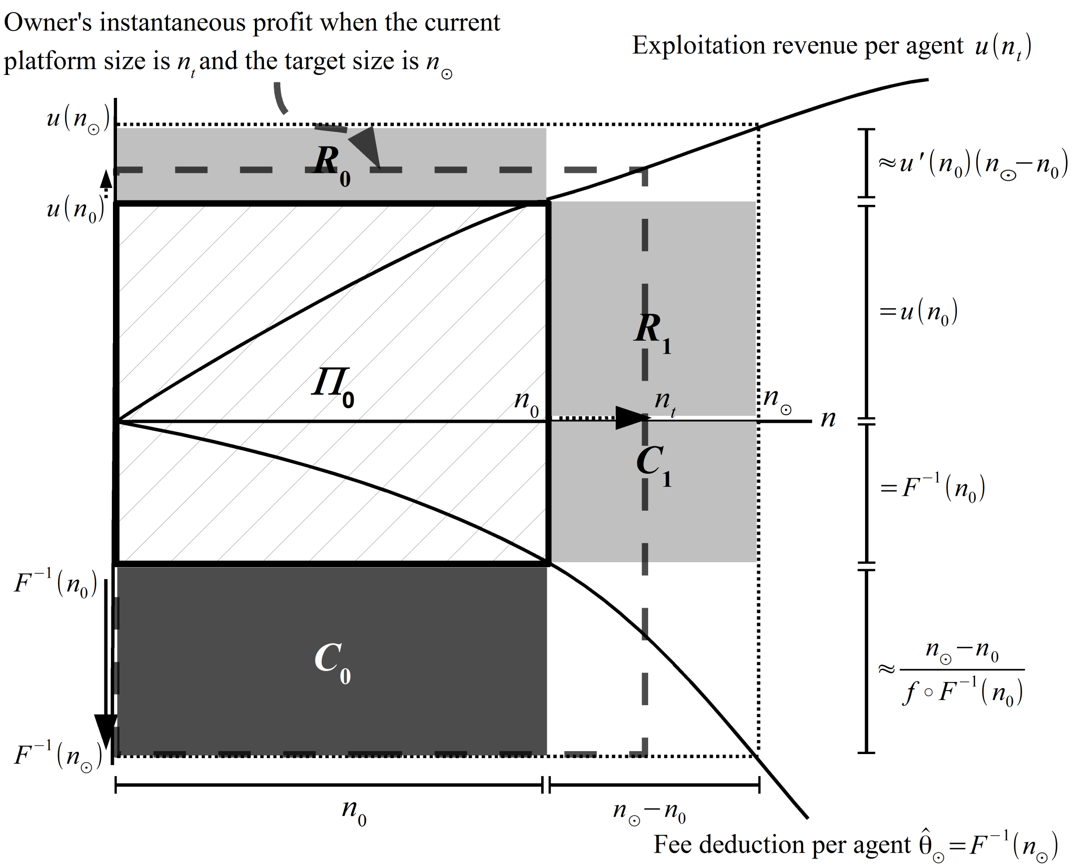

Figure 2: Breakdown of the platform owner's instantaneous profit into exploitation revenues and fee deducions.

Then, we have

$$\breve{\Pi}_\rho(n_0; n_0) = \widetilde{\Pi}_\rho(n_0) = \frac{1}{\rho}\widehat{\Pi}(n_0), \qquad \frac{\partial \breve{\Pi}}{\partial n_\odot}(n_0; n_0) = \frac{1}{1+\rho}\widetilde{\Pi}'_\rho(n_0).$$

Proof. See Section 5. □

We can identify the source of the "adjustment costs" in $\widetilde{\Pi}_\rho$ from the construction of $\breve{\Pi}_\rho$. Suppose that, while initial platform size n_0 was in a Nash equilibrium in the static entry game given $\hat{\theta}_0 := F^{-1}(n_0)$ before time 0, the owner sets larger target size $n_\odot > n_0$ and thus higher target type $\hat{\theta}_\odot > \hat{\theta}_0$ since time 0. The per-agent fee deduction increases to $\hat{\theta}_\odot$ *immediately* after this policy change, i.e., *at time 0*. The aggregate increase in the fee deductions for incumbent participants n_0 is indicated by C_0 in Figure 2.

In contrast, a participant's gross utility $u(n_t)$ is determined by *current* platform size n_t. Thus, the aggregate *exploitation revenue* $u(n_t)n_t$ increases *gradually* with the current size, not immediately with the target size. First, as the platform grows, new participants $n_t - n_0$ yield additional exploitation revenues R_1 for the platform owner, while there are also fee deductions C_1 for them. Hence, the net profit directly earned from new participants $R_1 - C_1$ is realized gradually with growth of the platform. Second, by positive network externality, *each incumbent participant's* gross utility increases as the platform size grows. This lets the owner earn more revenue from incumbent participants n_0. The revenue R_0 is realized gradually at the same speed as the realization of $R_1 - C_1$.

If the platform size jumps to new equilibrium $n_\odot = F(\theta_\odot)$ at the moment of the policy change, all of these changes take place immediately. Then, C_0 and $R_1 - C_1 + R_0$ are equally discounted by ρ; the discounted sum of these terms equals to

$$\begin{aligned}
&\frac{1}{\rho}\left[\overbrace{\{F^{-1}(n_\odot) - F^{-1}(n_0)\}}^{C_0} + \overbrace{\{u(n_0) - F^{-1}(n_0)\}(n_\odot - n_0)}^{R_1 - C_1} + \overbrace{\{u(n_\odot) - u(n_0)\}n_0}^{R_0}\right] \\
\approx&\frac{1}{\rho}\left[\frac{n_\odot - n_0}{f \circ F^{-1}(n_0)}n_0 + \{u(n_0) - F^{-1}(n_0)\}(n_\odot - n_0) + u'(n_0)\cdot(n_\odot - n_0)n_0\}\right] \\
=&\frac{1}{\rho}\widehat{\Pi}'(n_0)\cdot(n_\odot - n_0).
\end{aligned}$$

In contrast, under the BRD, the platform size changes toward the target size only gradually. The additional profit $R_1 - C_1 + R_0$ is realized so slowly that it is discounted by $\rho(1+\rho)$.[5] On the other hand, the fee deductions for incumbent participants C_0 are realized at time 0 and then discounted by the original discount rate ρ. The discounted sum puts more weight on C_0, like $\widetilde{\Pi}'_\rho$:

$$\begin{aligned}
&\frac{1}{\rho}\overbrace{\{F^{-1}(n_\odot) - F^{-1}(n_0)\}}^{C_0} + \frac{1}{\rho(1+\rho)}\left[\overbrace{\{u(n_0) - F^{-1}(n_0)\}(n_\odot - n_0)}^{R_1 - C_1} + \overbrace{\{u(n_\odot) - u(n_0)\}n_0}^{R_0}\right] \\
\approx&\frac{1}{\rho}\frac{n_\odot - n_0}{f \circ F^{-1}(n_0)}n_0 + \frac{1}{\rho(1+\rho)}\left[\{u(n_0) - F^{-1}(n_0)\}(n_\odot - n_0) + u'(n_0)\cdot(n_\odot - n_0)n_0\}\right] \\
=&\frac{1}{1+\rho}\widetilde{\Pi}'_\rho(n_0)\cdot(n_\odot - n_0).
\end{aligned}$$

In short, inertia in agents' decisions delays realization of the exploitation revenues, while the amount of fee deductions changes immediately after change in the target type. The loss of the exploitation revenues due to delayed realization is interpreted as the adjustment costs, which cause the platform size to shrink from the statically optimal size to the stationary size under the dynamically optimal pricing.

Theorem 1 implies the equivalence among the maximum of $\check{\Pi}_\rho$, that of $\widetilde{\Pi}_\rho$, and the stationary point under the dynamically optimal pricing. Further, it suggests that even if the owner is myopic and not fully rational in the sense that she does not take into account future change in the target size, she adjusts the target size in the same direction as under the dynamically optimal pricing: the myopic owner chooses to expand the platform if and only if the fully rational one also chooses expansion. Although the exact choice of the target size and thus the convergence speed may be different, myopic but recurrent adjustment under the fixed target fee scheme eventually results in the same stationary size in the long run as the dynamically optimal pricing. While it is beyond the scope of this paper to rigorously formulate such a myopic pricing dynamic, we state it as a less formal proposition.

[5] As $n_t - n_0 = (1 - e^{-t})(n_\odot - n_0)$, the realized additional profit is $(1 - e^{-t})(R_1 - C_1 + R_0)$ at time t. Its discounted sum is $\int_0^\infty e^{-\rho t}(1 - e^{-t})(R_1 - C_1 + R_0)dt = (R_1 - C_1 + R_0)/\{\rho(1+\rho)\}$.

Corollary 1. *For any discount rate $\rho > 0$ and platform size n_0, the following statements are equivalent.*

1. *Given initial platform size n_0, target size $n_\odot = n_0$ maximizes the discounted total profit $\check{\Pi}_\rho$ under the fixed target fee scheme.*

2. *The Lyapunov function $\widetilde{\Pi}_\rho$ is maximized at n_0.*

3. *n_0 is the platform size in the stationary state under the dynamically optimal pricing.*

Proposition 1. *Consider a myopic pricing dynamic where the owner occasionally revises the target size to optimize the fixed target fee scheme given the platform size at the moment of revision. The platform size converges to the same platform size as in the dynamically optimal pricing.*

Remark. One might argue that $\partial\check{\Pi}_\rho/\partial n_\odot$ is only proportional to $\widetilde{\Pi}'_\rho$ and they do not coincide exactly. To fill the gap, define $\bar{\Delta}\psi : \mathbb{R} \to \mathbb{R}$ as a function of $n_\odot$ by

$$\bar{\Delta}\psi(n_\odot) := \int_0^\infty e^{-\rho t}(\Psi_t(n_\odot; n_0) - n_0)dt, \qquad \text{given } n_0.$$

This $\bar{\Delta}\psi(n_\odot)$ can be read as the discounted total change in the platform size under the fixed target fee scheme with target size $n_\odot$. It reduces to $(n_\odot - n_0)/(1+\rho)$ by (4). Let $\overline{\Pi}_\rho(\bar{\Delta}n; n_0)$ be the discounted total profit under the fixed target fee scheme in which the discounted total change is $\bar{\Delta}n$ and the initial size is n_0:

$$\overline{\Pi}_\rho(\bar{\Delta}n; n_0) := \check{\Pi}_\rho(\bar{\Delta}\psi^{-1}(\bar{\Delta}n); n_0) = \check{\Pi}_\rho(n_0 + (1+\rho)\bar{\Delta}n; n_0).$$

Then, by chain rule, we find that $\widetilde{\Pi}_\rho$ provides a first-order approximation of $\overline{\Pi}_\rho(\cdot; n_0)$ in the following sense:

$$\overline{\Pi}_\rho(\bar{\Delta}n; n_0) \approx \widetilde{\Pi}_\rho(n_0) + \widetilde{\Pi}'_\rho(n_0)\bar{\Delta}n \qquad \text{if } \bar{\Delta}n \approx 0.$$

Corollary 2. *For each n_0, we have*

$$\overline{\Pi}_\rho(0; n_0) = \check{\Pi}_\rho(n_0; n_0) = \widetilde{\Pi}_\rho(n_0), \quad \frac{\partial\overline{\Pi}_\rho}{\partial\bar{\Delta}n}(0; n_0) = (1+\rho)\frac{\partial\check{\Pi}_\rho}{\partial n_\odot}(n_0; n_0) = \widetilde{\Pi}'_\rho(n_0).$$

5 Proof of Theorem 1

Proof. Let $\Delta n := n_\odot - n_0$. By linear approximations of V and F^{-1} around n_0, we obtain

$$V(n_t) = V(n_0) + V'(n_0)(n_t - n_0) + O_V(n_t - n_0),$$
$$\hat{\theta}_\odot = F^{-1}(n_\odot) = F^{-1}(n_0) + \frac{\Delta n}{f \circ F^{-1}(n_0)} + O_{\hat{\theta}}(\Delta n),$$

where $O_V(\Delta n)/\Delta n, O_{\hat{\theta}}(\Delta n)/\Delta n \to 0$ as $\Delta n \to 0$. The owner's instantaneous profit at time t is thus approximated as

$$\begin{aligned} P_t n_t =& V(n_t) - \hat{\theta}_\odot n_t \\ =& \left\{V(n_0) + V'(n_0)(1-e^{-t})\Delta n + O_V((1-e^{-t})\Delta n)\right\} \end{aligned}$$

$$
\begin{aligned}
&-\left\{F^{-1}(n_0)+\frac{\Delta n}{f\circ F^{-1}(n_0)}+O_{\hat\theta}(\Delta n)\right\}(n_0+(1-e^{-t})\Delta n)\\
=&\widehat{\Pi}(n_0)+\{V'(n_0)-F^{-1}(n_0)\}(1-e^{-t})\Delta n+\frac{n_0\Delta n}{h\circ H^{-1}(n_0)}+O_t(\Delta n),
\end{aligned}
$$

where $O_t(\Delta n)/\Delta n \to 0$ as $\Delta n \to 0$ for each t.

Hence the discounted sum of the owner's profits in the entire period $(0,\infty)$ is

$$
\begin{aligned}
&\breve{\Pi}_\rho(n_0+\Delta n;n_0)\\
=&\int_0^\infty e^{-\rho t}P_t n_t dt\\
=&\int_0^\infty e^{-\rho t}\left[\widehat{\Pi}(n_0)+\{V'(n_0)-F^{-1}(n_0)\}(1-e^{-t})\Delta n+\frac{n_0\Delta n}{f\circ F^{-1}(n_0)}+O_t(\Delta n)\right]dt\\
=&\frac{\widehat{\Pi}(n_0)}{\rho}+\left\{V'(n_0)-F^{-1}(n_0)\right\}\frac{\Delta n}{\rho(1+\rho)}-\frac{n_0}{f\circ F^{-1}(n_0)}\frac{\Delta n}{\rho}+\int_0^\infty e^{-\rho t}O_t(\Delta n)dt.
\end{aligned}
$$

It is immediate from this to see $\breve{\Pi}_\rho(n_0;n_0)=\widehat{\Pi}(n_0)/\rho$. The partial derivative of $\breve{\Pi}_\rho$ with respect to $n_\odot$ at $(n_\odot,n_0)=(n_0,n_0)$ is

$$
\begin{aligned}
\frac{\partial\breve{\Pi}_\rho}{\partial n_\odot}(n_0;n_0)=&\lim_{\Delta n\to 0}\frac{\breve{\Pi}_\rho(n_0+\Delta n;n_0)-\breve{\Pi}_\rho(n_0;n_0)}{\Delta n}\\
=&\left\{V'(n_0)-F^{-1}(n_0)\right\}\frac{1}{\rho(1+\rho)}-\frac{n_0}{f\circ F^{-1}(n_0)}\frac{1}{\rho}+\int_0^\infty e^{-\rho t}\left(\lim_{\Delta n\to 0}\frac{O_t(\Delta n)}{\Delta n}\right)dt\\
=&\frac{1}{1+\rho}\widetilde{\Pi}'_\rho(n_0).
\end{aligned}
$$

□

Declarations

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