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Lyapunov's direct method for stability of a set and its transitivity under a differential inclusion*

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Abstract

We present a version of Lyapunov's direct method for stability of a set under a differential inclusion. We pay careful attention to the assumption of forward invariance of a basin of attraction, which is often overlooked when applying the method to local stability. Even if the value of a local Lyapunoov function monotonically changes in some neighborhood of the limit set, this alone does not prevent a trajectory from escaping from this particular neighborhood. In this note, we verify that we can construct a smaller but forward invariant neighborhood. As a corollary, we obtain a transitivity theorem on basins of attractions without requiring forward invariance.

Keywords: Lyapunov function, stability of a set, forward invariance, evolutionary dynamics.

1 Introduction

We present a version of Lyapunov's direct method for stability of a set under a differential inclusion. Specifically, our version in this paper differs from known versions for differential inclusions as i) the limit can be a set, not necessarily a single point as in Smirnov (2001); Bacciotti and Ceragioli (1999) and ii) a sufficient condition is based on only local information (derivative with respect to state variable) of a Lyapunov function at each *point* around the limit, not requiring information about intertemporal changes in the value of the Lyapunov function or its *time* derivative as in Benaïm et al. (2005). Our first motivation for this note is to serve as a handy reference for applications by reducing the burden of proof and making these points explicit. Our version is particularly useful for a practitioner (say, a game theorist) to prove local stability of a set when the basin of

*A chat with Bill Sandholm at Stony Book in 2016 led me to paying attention to the assumption of invariance and to realizing the merits of theorems in this paper. (I still remember the chat at a parking lot of Hilton Garden Inn after our lunch.) As well as uncountably many inputs and encouragements, I owe so much to Bill.

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attraction, which must be forward invariant, is not known for the practitioner. Furthermore, this allows us to verify transitivity of stability of sets from Lyapunov functions.

The idea of Lyapunov stability theorem or Lyapunov's direct method appeals to intuition: if we find a mapping (*Lyapunov function*) from the current state of a dynamic to a real number such that i) the function attains a local minimum only at an equilibrium (possibly a set) and ii) its value decreases as long as the current state has not reached the equilibrium, then the equilibrium is stable under the dynamic. With this idea on hand, we hope to tell local stability without identifying a solution path; we just find a Lyapunov function and see how its value changes in a neighborhood of an equilibrium. So, we typically find a neighborhood where the decrease in the Lyapunov function is guaranteed, which we call here a *monotone decreasing neighborhood*; one might expect this neighborhood to be a basin of attraction.

However, a basin of attraction must be forward invariant. (This does not matter for global stability, of course.) Even if we find a monotone decreasing neighborhood, a solution path may escape from this neighborhood and eventually the Lyapunov function may not decrease after the escape. This imposes an additional burden of proof, losing an appeal of the theorem to intuition since we eventually need to identify a solution path.

Similarly, we would expect transitivity of stable sets based on Lyapunov functions. That is, if we find a Lyapunov function that decreases in X_1 and attains the minimum in X_2 and another that decreases in X_2 and attains the minimum in X^*, then we expect X^* to be stable in X_1. Again, the known version of the transitivity theorems as in Conley (1978) requires X_1 to be forward invariant and X_2 to be forward and also strongly negative (i.e., backward) invariant.[1]

In applications to economics or game theory, we hope to find a Lyapunov function from economic intuition. Under an evolutionary dynamic in a game, an aggregate of agents' possible gains from adjustment of their choices can be used as a candidate for a Lyapunov function once we find a neighborhood where an agent's revision of the choice incurs negative payoff externality to others' gains from further changes; generality of this proof strategy is verified by Zusai (2020). This neighborhood is a monotone decreasing neighborhood. However, forward invariance needs more mathematical examination of the dynamic system itself, which may not be appealing to economic intuition.

In this paper, we reduce the burden of proof by showing that we can construct a forward invariant (smaller) neighborhood from a monotone decreasing neighborhood. This is a unique issue for *local* stability and has been overlooked when one applies a theorem on global stability to local stability by analogy; as far as the author finds, this issue is also not explicitly solved in known proofs as in Smirnov (2001). Our proof fixes this hole. Further, this helps us to establish a transitivity theorem without requiring forward or negative invariance.

We consider a differential inclusion (a set-valued differential equation) and conver-

[1] Strong negative invariance of X means that, if a solution path (starting at time 0) visits X at any time $t \geq 0$, then it must have started from X at time 0.

gence to an equilibrium set, not necessarily a point. This generalization is needed to cover evolutionary dynamics in games, since Nash equilibrium may constitute a (connected) set and also a transition vector may not be uniquely specified when there are multiple best responses.

In the next section, we present our theorems with quick summary of necessary notation and concepts. We discuss applications in evolutionary game theory in Section 3. In Section 4, this leads to comparison of our theorems with those in preceding literature to highlight significance of our theorem, exemplified in those applications. The proofs are given in Section 5.

2 Definitions and theorems

We consider an autonomous differential inclusion $\boldsymbol{\mathcal{V}} : \mathcal{X} \rightrightarrows T\mathcal{X}$ on a compact metric A-dimensional real space $\mathcal{X} \subset \mathbb{R}^A$ with $A < \infty$, such as

$$\dot{\mathbf{x}} \in \boldsymbol{\mathcal{V}}(\mathbf{x}) \qquad \text{at each } \mathbf{x} \in \mathcal{X}.$$

$T\mathcal{X}$ stands for the tangent space of $\mathcal{X}$.[2] As a solution concept for the differential inclusion, we adopt a Carathéodory solution; that is, a solution trajectory $\{\mathbf{x}_t\}_{t\geq 0}$ must be Lipschitz continuous at every $t \geq 0$ and also differentiable with derivative $\dot{\mathbf{x}}_t \in \boldsymbol{\mathcal{V}}(\mathbf{x}_t)$ at almost every t. To guarantee the existence of a solution trajectory from an arbitrary point $\mathbf{x}_0 \in \mathcal{X}$, we assume that correspondence (set-valued mapping) $\boldsymbol{\mathcal{V}}$ is bounded and upper semicontinuous, with compact and convex values (Smirnov, 2001, Ch. 4).

Let X^* be a nonempty closed set. We say X^* is **Lyapunov stable** under $\boldsymbol{\mathcal{V}}$ if for any open neighborhood O of X^* there exists a neighborhood O' of A such that *every* solution path $\{\mathbf{x}^t\}_{t\geq 0}$ that starts from O' remains in O.[3] X^* is **attracting** if there is a neighborhood O of X^* such that *every* solution that starts in O converges to X^*; O is called a basin of attraction to X^*. If it is the entire space $\mathcal{X}$, then we say X^* is globally attracting. X^* is **asymptotically stable** if it is Lyapunov stable and attracting; it is globally asymptotically stable if it is Lyapunov stable and globally attracting. In this paper, we prove Theorem 1 and corollary 1.

Lyapunov's direct method

Theorem 1 (Lyapunov stability theorem)**.** *Let X^* be a non-empty closed set in a compact metric space $\mathcal{X} \subset \mathbb{R}^A$ with tangent space $T\mathcal{X}$, and X' be an open neighborhood of X^*. Suppose that continuous function $W : \mathcal{X} \to \mathbb{R}$ and lower semicontinuous function $\tilde{W} : \mathcal{X} \to \mathbb{R}$ satisfy (a) $W(\mathbf{x}) \geq 0$ and $\tilde{W}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in X'$ and (b)* $\operatorname{cl} X' \cap W^{-1}(0) = \operatorname{cl} X' \cap \tilde{W}^{-1}(0) = X^*$.

[2] Henceforth the definitions follow Sandholm (2010b).

[3] This allows a solution path to transit between points in X^*. It is indeed a case for the best response dynamic in a connected set of Nash equilibria.) Because of this, our set-wise stability notion is weaker than stability of each point in X^*.

In addition, assume that W is Lipschitz continuous in $\mathbf{x} \in X'$. If a differential inclusion $\boldsymbol{\mathcal{V}} : \mathcal{X} \to T\mathcal{X}$ satisfies[4]

$$DW(\mathbf{x})\dot{\mathbf{x}} \leq \tilde{W}(\mathbf{x}) \qquad \textit{for any } \dot{\mathbf{x}} \in \boldsymbol{\mathcal{V}}(\mathbf{x}) \tag{1}$$

whenever W is differentiable at $\mathbf{x} \in X'$, then X^* is asymptotically stable under $\boldsymbol{\mathcal{V}}$.

W is a **Lyapunov function**; we call $\tilde{W}$ a **decaying rate function** and X' a **monotone decreasing neighborhood**. Note that we allow for multiplicity of transition vectors, while requiring functions W and $\tilde{W}$ to be well defined (the uniqueness of the values) as functions of state variable $\mathbf{x}$, independently of the choice of transition vector $\dot{\mathbf{x}}$ from $\boldsymbol{\mathcal{V}}(\mathbf{x})$.

Generally speaking, in Lyapunov's direct method, monotone decrease in the value of the Lyapunov function W along with a solution trajectory to zero implies convergence to zeros of W. In our version, condition (1) guarantees monotone decrease along with any solution path starting from some neighborhood of X^*, possibly smaller than X', only based on local information on the derivative of W at each state $\mathbf{x}$ without requiring explicit identification of the solution trajectory. Notice that we do not assume forward invariance of X'. Thus, it is possible that a solution trajectory may escape from X' and then W may eventually increase; see Fig. 1. Therefore, X' itself may not be a basin of attraction to X^*. Yet, our theorem guarantees asymptotic stability of X^* since we can construct a forward invariant set from a subset of X'.

In a standard Lyapunov stability theorem (e.g. Robinson (1998, §5.5.3)) for a differential equation, a decaying rate function $\tilde{W}$ is not explicitly required while $\dot{W}$ is assumed to be (strictly) negative until $\mathbf{x}$ reaches the limit set X^*. The most significant difference is the requirement of lower semicontinuity of $\tilde{W}$. This assures the existence of a lower bound on the decaying rate $\dot{W}(\mathbf{x}) \leq \bar{w} < 0$ over any solution trajectories in a hypothetical case in which $\mathbf{x}$ remained out of an arbitrarily small neighborhood of X^* for an arbitrarily long period of time. This excludes the possibility that $\mathbf{x}$ would stay out there forever and guarantees convergence to X^* (not only Lyapunov stability, i.e., no asymptotic escape from X^*).

Lyapunov-based transitivity theorem.

Corollary 1 (Transitivity theorem). *Let X_1, X_2, and X^* be three non-empty subsets of a compact metric space $\mathcal{X}$ such that $X_1 \supset X_2 \supset X^*$; assume that X^* is closed and X_1 is open. Suppose that two Lipschitz continuous functions $W_1, W_2 : \mathcal{X} \to \mathbb{R}$ and two lower semicontinuous functions $\tilde{W}_1, \tilde{W}_2 : \mathcal{X} \to \mathbb{R}$ satisfy the following assumptions: for any $\mathbf{x} \in X_1$,*

a) i) $W_1(\mathbf{x}) \geq 0$, ii) $\tilde{W}_1(\mathbf{x}) \leq 0$, and iii) $\mathrm{cl}\, X_1 \cap W_1{}^{-1}(0) = \mathrm{cl}\, X_1 \cap \tilde{W}_1^{-1}(0) = \mathrm{cl}\, X_2$;

b) i) $W_2(\mathbf{x}) \geq 0$, ii) $\left[\mathbf{x} \in X_2 \ \Rightarrow\ \tilde{W}_2(\mathbf{x}) \leq 0\right]$, and iii) $\mathrm{cl}\, X_2 \cap W_2{}^{-1}(0) = \mathrm{cl}\, X_2 \cap \tilde{W}_2^{-1}(0) = X^*$;

[4] D denotes differentiation, so $DW(\mathbf{x}) = dW/d\mathbf{x}(\mathbf{x}) = [\partial W/\partial x_1(\mathbf{x}), \ldots, \partial W/\partial x_A(\mathbf{x})]$.

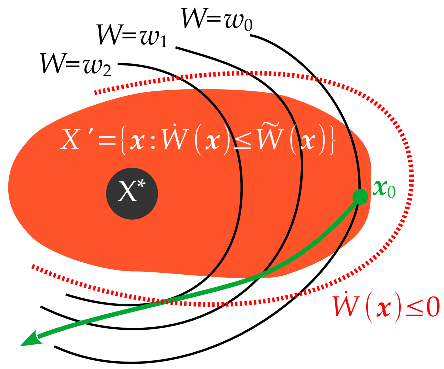

Figure 1: A worrisome situation. Here $w_0 > w_1 > w_2$ and $\dot{W}$ is negative in the area within the red dashed curve (inside toward X^*); X' is contained in this area. Along a green solution trajectory, the value of W decreases until it crosses the red curve but increases after it and escapes from X^*. Notice that, without forward invariance of X', the solution trajectory leaves it while W is still decreasing.

c) $\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x}) \leq 0$.

Furthermore, assume that

$$\text{a-iv) } DW_1(\mathbf{x})\dot{\mathbf{x}} \leq \tilde{W}_1(\mathbf{x}), \qquad \text{b-iv) } DW_2(\mathbf{x})\dot{\mathbf{x}} \leq \tilde{W}_2(\mathbf{x}) \qquad \text{for any } \dot{\mathbf{x}} \in \boldsymbol{\mathcal{V}}(\mathbf{x}),$$

whenever W_1 and W_2 are differentiable at $\mathbf{x} \in X_1$. Then, X^* is asymptotically stable under $\boldsymbol{\mathcal{V}}$.

One might expect the set of conditions in b) to imply asymptotic stability of X^* thanks to Theorem 1. But here X_2 may not be an open set. In particular, as we will see in Section 3, we want to cover a case where X_2 is just a hyperplane in $\mathcal{X}$ as in Figure 2; for example on evolution in a population game, X^* can be a set of Nash equilibria and X_2 is a set of states where strictly dominated strategies are not used. Then, X_2 is not open. Of course, X_2 may not be invariant; indeed, the known transitivity theorem as in Conley (1978) would require an interim set X_2 to be both forward and backward (negative) invariant. However, to appeal to intuition, a practitioner may hope to establish asymptotic stability of X^* by first constructing a Lyapunov function on X_1 for stability of X_2 and then another that would work on the hyperplane X_2 .

We fix this gap by imposing c). Since we assume openness of X_1 in a), the set of conditions in a) indeed implies asymptotic stability of $\operatorname{cl} X_2$. W_1 stops decreasing once a solution trajectory reaches X_2, while W_2 may have increased before that. So, either one of them alone is not enough to serve as a Lyapunov function for convergence to X^*. Condition c) essentially implies that a proper linear combination of these two functions resolves these two issues and serves as a Lyapunov function for the entire region.

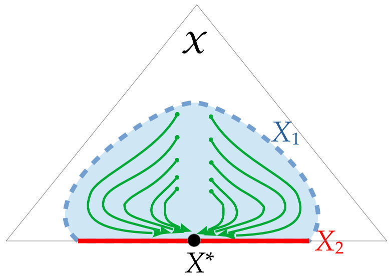

Figure 2: Pseudo-transitivity theorem. Here X^* is a singleton of a point on the line segment X_2. Notice that X_2 is not open in $\mathcal{X}$.

3 Applications in game theory

The merit of our theorems lies in allowing a practitioner to verify a Lyapunov function just by finding a neighborhood of the limit set where the value of the Lyapunov function decreases, without worry about possibilities that a solution trajectory might escape from this neighborhood. This reduces a burden of proof significantly and leads to an intuitively natural proof, as we discuss here.

Stability of an equilibrium has been a fundamental issue in economic theory and game theory. An economic model typically focuses on the payoff structure, i.e., how payoffs (returns of choices) are determined from choices of agents in the model. Economists argue that if payoffs negatively respond to perturbation of agents' choices from equilibrium, it should drive agents to returning to equilibrium. Economists regard such a negative feedback of payoffs as a (sufficient) condition for stability, calling *static stability*. By explicitly formulating agents' processes of revising their choices (that is, an evolutionary dynamic in economic sense, see Sandholm (2010b)), evolutionary game theorists have attempted to test if such a negative feedback of payoffs indeed assure asymptotic stability of an equilibrium.

Concrete formulation of static stability differs by specification of a game and the context of analysis. Here we specifically consider a population game in a strategic form, a predominant playground in evolutionary game theory. A population game consists of a large population of agents, each of which makes an independent decision. In a strategic form, an agent's decision is a choice of a "strategy" from a given strategy set, say $\mathcal{S} = \{1, \ldots, S\}$. Each strategy $s \in \mathcal{S}$ yields payoff $F_s(\mathbf{x}) \in \mathbb{R}$, depending on distribution of agents over the strategies $\mathbf{x} = (x_s)_{s\in\mathcal{S}} \in \Delta^S := \{\mathbf{x} = (x_s)_{s\in\mathcal{S}} \in \mathbb{R}^S \mid x_s \geq 0 \forall s \in \mathcal{S} \text{ and } \sum_s x_s = 1\}$. State $\mathbf{x}^* \in \Delta^S$ is a **Nash equilibrium** if $(\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{F}(\mathbf{x}^*) \leq 0$ for all $\mathbf{x} \in \Delta^S$. Static stability in this setting is conceptualized as a property of **contraction** of payoff function $\mathbf{F} = (F_s)_{s\in\mathcal{S}}$, namely negative semidefiniteness of its Jacobian $D\mathbf{F}$: $\mathbf{z} \cdot D\mathbf{F}(\mathbf{x})\mathbf{z} \leq 0$ for all $\mathbf{z} \in \mathbb{R}^S$.. A game is called a **contractive game** if contraction property holds globally, i.e., at any $\mathbf{x} \in \Delta^S$.

While it is proven for each of canonical evolutionary dynamics such as the best response dynamic and excess payoff dynamics that contraction guarantees asymptotic stability of the set of Nash equilibria (Hofbauer and Sandholm, 2009), Zusai (2020) presents a unifying general proof that covers any evolutionary dynamic as long as agents' revision of strategies is "rationalizable", i.e., a new strategy after each revision can be explained as a constrained optimal choice, albeit the new strategy may not maximize the payoff, by introducing a (possibly stochastic) switching cost to rationalize reluctance to switch and also a (possibly stochastic) restriction to available strategies to rationalize a switch to a suboptimal strategy. Then, Zusai (2020) defines a ***net* gain** from revision by subtracting the switching cost from payoff improvement by the switch and finds that the aggregate net gain G attains zeros (only) at Nash equilibria and also it is coupled with a negative definite function $H : \Delta^S \to \mathbb{R}$ such that[5]

$$DG(\mathbf{x})\dot{\mathbf{x}} = H(\mathbf{x}) + \dot{\mathbf{x}} \cdot DF(\mathbf{x})\dot{\mathbf{x}} \qquad \text{for any } \mathbf{x} \in \Delta^S, \dot{\mathbf{x}} \in \mathbf{V}(\mathbf{x}).$$

Contraction implies $DG(\mathbf{x})\dot{\mathbf{x}} \leq H(\mathbf{x})$, so we can apply Theorem 1 to G as a Lyapunov function and H as a decay rate function. Our theorem covers both a global case (i.e., a contraction game) and a local case in which the contraction property holds only in some neighborhood of an equilibrium set. So, this argument provides a single proof for global stability of the set of Nash equilibria in the former case and local stability of a Nash equilibrium in the latter case. The neighborhood for local contraction is identified only from information about the game, especially, $\mathbf{F}$, though a forward invariant neighborhood needs identification of a solution trajectory.

Speaking about local stability in a game, an evolutionary stable state (ESS) is the classical static stability concept in particular to evolutionary game theory. While the contraction property is locally satisfied at an ESS (or its weaker version, a neutrally stable state) if it lies in the interior of Δ^S, a weaker version of contraction has been considered to cover the boundary case Taylor and Jonker (1978): $\mathbf{x}$ is called a **regular ESS** if 1)

$$F_u(\mathbf{x}^*) < \max\{F_s(\mathbf{x}^*) \mid s \in \mathcal{S}\} \quad \text{for any } u \in U^* := \{s \in \mathcal{S} \mid x_s^* = 0\}.$$

and 2) there exists a neighborhood of $\mathbf{x}^*$ in which each point $\mathbf{x}$ satisfies

$$\mathbf{z} \cdot D\mathbf{F}(\mathbf{x})\mathbf{z} \leq 0 \qquad \text{for any } \mathbf{z} \in \mathbb{R}^S_{U^*} := \{\mathbf{z} \in \mathbb{R}^S \mid x_u^* = 0 \Rightarrow z_u = 0\}.$$

Here U^* is the set of strategies that are unused at $\mathbf{x}^*$. 1) means that none of those strategies is indeed optimal at $\mathbf{x}^*$. Condition 2) essentially means contraction but it is weakened to require the contraction only when those strategies in U^* are kept unused.

Sandholm (2010a) suggests for pairwise comparison dynamics and the standard best response dynamic that, once we can somehow obtain a Lyapunov function for a contractive game, we can use it to construct that for a regular ESS. To fill the gap by weakening the requirement of contraction, he suggests to add (a sufficiently large scalar multiple of) the mass of players of those strategies in U^*, i.e., $M_{U^*}(\mathbf{x}) := \sum_{u \in U^*} x_u$, to the Lyapunov

[5]See Zusai (2020) for an exact definition of G.

function. By applying Corollary 1 to the combination of G in Zusai (2020) and M_{U^*} in Sandholm (2010a), we can generalize dynamic stability of a regular ESS to a wide class of rationalizable evolutionary dynamics.

4 Literature review

Lyapunov's direct method

Here we clarify the merit of our own version of the Lyapunov's direct method theorem from comparison with preceding versions of the theorem.

As we have repeatedly stated, the technical difference lies in not requiring the prior knowledge on a forward invariance neighborhood. Actually this issue has been overlooked. Lyapunov's direct method is typically proven for global stability (i.e., condition (1) holds globally); see Bacciotti and Ceragioli (1999) and Clarke et al. (1998, Theorem 5.5) for global stability of a single point. Then, we easily expect its analogous result to hold for a local case. It is not trivial as we discussed and highlighted.—It is indeed a matter when trying to use a transitivity theorem.

When local stability of a set is explicitly argued, a similar condition such that

$$W(\mathbf{x}_t) \leq W(\mathbf{x}_{t'}) \quad \text{whenever } t \geq t' \geq 0 \text{ and } \mathbf{x}_0 \in X \tag{2}$$

or

$$\dot{W}(\mathbf{x}_t) \leq \tilde{W}(\mathbf{x}_t) \quad \text{at any } t \geq 0 \text{ whenever } \mathbf{x}_0 \in X \tag{3}$$

is imposed; see Benaïm et al. (2005, Prop. 3.25) and Zusai (2018, Theorem 7). If X' is forward invariant, then (1) implies these conditions. The current version weakens the assumption of the forward invariance and thus applies to cases in which the neighborhood X is not known *a priori*. Yet, we retrieve asymptotic stability of X^* by obtaining a forward invariant subset of X'.

It may be possible to restrict the domain of W and $\tilde{W}$ to X'. Then, one might easily conclude that, with some constant $\bar{w}$, a lower contour set $W^{-1}([0, \bar{w}])$ should serve as a forward invariant set since condition (1) would imply that W cannot increase as in (2) and (3); see Hale (1980, pp.313–4) for a proof for local stability of an isolated equilibrium point under a differential equation and Smirnov (2001, Theorem 8.2) for that under a differential inclusion, both presuming that this argument works. However, this argument presumes that condition (1)always holds along with a solution trajectory to derive (3) from (1) as argued above. This argument still misses the possibility that a trajectory starting from X' may escape from X'. We are questioning this presumption and fix the hole, while also extending pointwise stability as in Smirnov (2001) to set-wise stability.

Transitivity theorem

One might (wrongly) argue that conditions a) and b) in Corollary 1 imply asymptotic stability of X_2 with X_1 as a monotone decreasing neighborhood and that of X^* with X_2, by applying only Theorem 1. From this chain relationship, one might want to use the standard transitivity theorem as in Conley (1978) as we argued in Section 1. However, Conley's theorem requires both forward and strongly negatively invariance of interim set X_2. However, a monotone decreasing neighborhood may not be invariant and thus we cannot apply Conley's theorem. Furthermore, we do not assume openness of the interim set X_2; in the application to a regular ESS, it is indeed a hyperplane defined by $\sum_{u \in U^*} x_u = 0$. The openness of monotone decreasing set X' is crucial for our Theorem 1, since we eventually construct a smaller forward invariant set within the monotone decreasing set.[6] So, we cannot apply Theorem 1 to obtain asymptotic stability of X^* from condition b). We resolve these issues by reconstructing a Lyapunov function.

5 Proofs

Proof of Theorem 1

Proof. Here we prove the difference from Zusai (2018, Theorem 7), which assumes (3). For this, we focus on the case of $X' \subsetneq \mathcal{X}$ and find a forward invariant subset of X'. Once we find it, any Carathéodory solution starting from the forward invariant subset remains there and thus satisfies (3) as (1) holds for $\mathbf{x} = \mathbf{x}_t$ at each time $t \in \mathbb{R}_+$. Then, Zusai (2018, Theorem 7) is applied and assures asymptotic stability of X^* while having the forward invariant subset as a basin of attraction.

First, construct a distance from point $\mathbf{x} \in \mathcal{X}$ to X^* based on the metric on $\mathcal{X}$, say $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, by

$$d_*(\mathbf{x}) := \min_{\mathbf{x}^* \in X^*} d(\mathbf{x}, \mathbf{x}^*).$$

Since X^* is a non-empty compact set and $d(\mathbf{x}, \mathbf{x}^*)$ is continuous in $\mathbf{x}^*$ when $\mathbf{x}$ is fixed, Weierstrass theorem assures the existence of the minimum in the above definition of $d_*(\mathbf{x})$. This d_* satisfies

$$d_*(\mathbf{x}) \geq 0; \qquad d_*(\mathbf{x}) = 0 \;\Leftrightarrow\; \mathbf{x} \in X^*.$$

Let $\bar{d}$ be the shortest distance from the complement of X' to X^*:

$$\bar{d} := \min_{\mathbf{x} \in \mathcal{X} \setminus X'} d_*(\mathbf{x}). \tag{4}$$

[6] Specifically, in our proof, we use openness of monotone decreasing neighborhood X' to confirm the existence and (nonzero) positiveness of $\bar{d}$, i.e., the shortest distance from X^* to get out of X'. In case X' is not open, we could define $\bar{d}$ by replacing minimum with infimum. But this may be zero especially if X' is only a hyperplane.

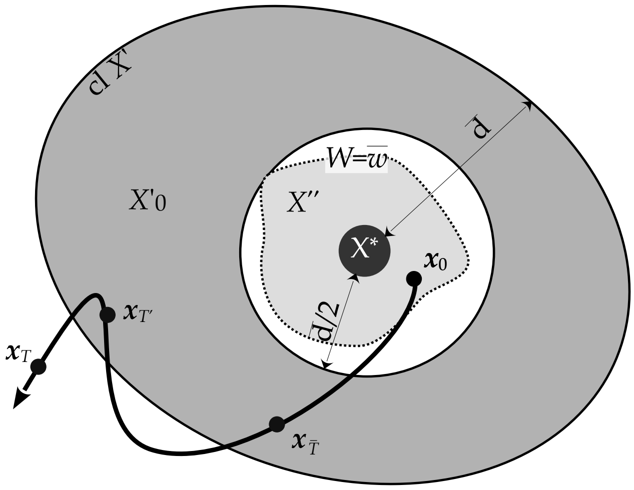

Figure 3: Sets in the proof of Theorem 1. X^* is the black area in the center and X'' is the light gray area. $\mathrm{cl}\, X'$ is the entire oval, with the outermost outline. X'_0 is the dark gray area, including the both boundaries.

Maximum theorem guarantees continuity of $d_* : \mathcal{X} \to \mathbb{R}_+$ by continuity of $d(\mathbf{x}, \mathbf{x}^*)$ in both $\mathbf{x}$ and $\mathbf{x}^*$. Besides, $\mathcal{X} \setminus X'$ is a non-empty compact subset by $X' \subsetneq \mathcal{X}$ and the openness of X'. Hence, the minimum in (4) exists. It follows that

$$\bar{d} > 0; \qquad d_*(\mathbf{x}) < \bar{d} \;\Rightarrow\; \mathbf{x} \in X'. \tag{5}$$

Define set $X'_0 \subset \mathrm{cl}\, X'$ by

$$X'_0 := \mathrm{cl}\, X' \,\cap\, d_*{}^{-1}([\bar{d}/2, \infty)).$$

Since both $\mathrm{cl}\, X'$ and $d_*{}^{-1}([\bar{d}/2, \infty))$ are closed, X'_0 is closed and thus compact in $\mathcal{X}$. It is not empty, as proven here. Suppose $X'_0 = \emptyset$; then, any $\mathbf{x} \in \mathcal{X}$ with $d_*(\mathbf{x}) \geq \bar{d}/2$ must be out of $\mathrm{cl}\, X'$. On the other hand, since $\mathrm{cl}\, X'$ is not empty, X' has at least one boundary point $\mathbf{x}^0$; then, $d_*(\mathbf{x}^0) \geq \bar{d}$.[7] By the former statement, this implies $\mathbf{x}^0 \notin \mathrm{cl}\, X'$ but it contradicts with $\mathbf{x}^0$ being on the boundary of X'; hence, X'_0 cannot be empty.

Let $\bar{w}$ be the minimum of W in X'_0;

$$\bar{w} := \min_{\mathbf{x} \in X'_0} W(\mathbf{x}).$$

Since X'_0 is compact and nonempty and W is (Lipschitz) continuous, the minimum exists. Furthermore, it is positive; we have $X'_0 \subset \mathrm{cl}\, X'$ by construction and $W(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathrm{cl}\, X'$ by condition (a) and continuity of W, while no element $\mathbf{x} \in X'_0$ belongs to X^* since $d_*(\mathbf{x}) \geq \bar{d} > 0$ for any $\mathbf{x} \in X'_0$. Because $X^* = \mathrm{cl}\, X' \cap W^{-1}(0)$ by condition (b) and $X'_0 \subset \mathrm{cl}\, X'$, this implies $\mathbf{x} \in X'_0 \Rightarrow W(\mathbf{x}) > 0$. Hence we have $\bar{w} > 0$. By the definition of

[7] We can make a sequence converging to $\mathbf{x}^0$ from elements out of X', whose distance from X^* cannot be smaller than $\bar{d}$ by (4).

$\bar{w}$, we have

$$\underbrace{\left[\mathbf{x} \in \operatorname{cl} X' \text{ and } d_*(\mathbf{x}) \geq \bar{d}/2\right]}_{\text{i.e., } \mathbf{x} \in X'_0} \Rightarrow W(\mathbf{x}) \geq \bar{w}. \tag{6}$$

Define set $X'' \subset X'$ by

$$X'' = W^{-1}([0, \bar{w}) \cap X'. \tag{7}$$

This set is an open neighborhood of X^* by $X^* \subset X''$, since $W = 0$ at anywhere in X^* and $X^* \subset X'$. Now we prove that X'' is wholly contained in set $d_*^{-1}([0, \bar{d}/2))$. Assume that there exists $\mathbf{x} \in X''$ such that $d_*(\mathbf{x}) \geq \bar{d}/2$. These jointly imply $W(\mathbf{x}) \geq \bar{w}$ by (6) since $\mathbf{x} \in X'' \subset X' \subset \operatorname{cl} X'$. However, this contradicts with $W(\mathbf{x}) \in [0, \bar{w})$ for $\mathbf{x}$ to belong to X''. Hence, we have

$$\mathbf{x} \in X'' \Rightarrow d_*(\mathbf{x}) < \bar{d}/2. \tag{8}$$

Now we prove X'' is forward invariant; remind that X' may not be forward invariant and thus it is not trivial to confirm that a trajectory from X'' stays in X'. To verify it by contradiction, assume that there is a Carathéodory solution trajectory $\{\mathbf{x}_t\}$ starting from X'' but escaping X'' at some moment of time:

$$\mathbf{x}_0 \in X'', \qquad \text{and } \mathbf{x}_T \notin X'' \text{ at some } T > 0. \tag{9}$$

The statement $\mathbf{x}_T \notin X''$ means $\mathbf{x}_T \notin X'$ or $W(\mathbf{x}_T) \geq \bar{w}$ by (7). In the former case, we have $d_*(\mathbf{x}_T) \geq \bar{d}$ by (4) while $d_*(\mathbf{x}_0) < \bar{d}/2$ by (8). By continuity of $d_*(\mathbf{x})$ in $\mathbf{x}$ and of $\mathbf{x}_t$ in t on a Carathéodory solution trajectory $\{\mathbf{x}_t\}$, $d_*(\mathbf{x}_t)$ is continuous in t; hence, there exists a moment of time $T' \in (0, T)$ such that $d_*(\mathbf{x}_{T'}) = 0.9\bar{d} \in (0.5\bar{d}, \bar{d}) \subset (d_*(\mathbf{x}_0), d_*(\mathbf{x}_T))$. At this point, $\mathbf{x}_{T'} \notin X''$ by (8) while $\mathbf{x}_{T'} \in X'$ by (5); thus, $W(\mathbf{x}_{T'}) \geq \bar{w}$ by (7). Hence, the first case of escaping X'' implies the existence of $T' > 0$ such that

$$W(\mathbf{x}_{T'}) \geq \bar{w} \qquad \textit{and} \qquad \mathbf{x}_{T'} \in X'.$$

In the second (but not the first) case, we have $W(\mathbf{x}_T) \geq \bar{w}$ but $\mathbf{x}_T \in X'$; that is, the above statement holds with $T' = T$.

The above two conditions on T' implies the existence of $\bar{T} \in (0, T']$ such that

$$W(\mathbf{x}_{\bar{T}}) \geq \bar{w}, \qquad \text{and} \qquad \left[\mathbf{x}_t \in X' \text{ for all } t < \bar{T}\right]. \tag{10}$$

To prove it, assume $\mathbf{x}_{t'} \notin X'$ at some $t' < T'$, i.e., the negation of the latter condition with $\bar{T} = T'$; if there is no such $t' \leq T'$, then it suggests that the claim (10) holds at $\bar{T} = T'$ by the fact $W(\mathbf{x}_{T'}) \geq \bar{w}$. By (5), the hypothesis $\mathbf{x}_{t'} \notin X'$ implies $d_*(\mathbf{x}_{t'}) \geq \bar{d}$. Again, by continuity of $d_*(\mathbf{x}_t)$ in t, the set $\{t \leq t' \mid d_*(\mathbf{x}_t) \geq \bar{d}\}$ is closed and thus compact. This implies the existence of the minimum $\bar{T}$ in this set, and further $\bar{T} > 0$ by the fact $d_*(\mathbf{x}_0) < \bar{d}/2$. That is, we have $d_*(\mathbf{x}_t) < \bar{d}$ for all $t < \bar{T}$ while $d_*(\mathbf{x}_{\bar{T}}) = \bar{d}$. The former implies $\mathbf{x}_t \in X'$ for all $t < \bar{T}$ by (5) and the latter implies $W(\mathbf{x}_{\bar{T}}) \geq \bar{w}$ by

$\mathbf{x}_{\bar{T}} = \lim_{t \to \bar{T}} x_t \in \mathrm{cl}\, X'$ and (6). Thus, the above claim (10) holds at this $\bar{T} \in (0, T']$.

Since condition (a) and (1) hold almost everywhere in X', we have $\dot{W}(\mathbf{x}_\tau) \leq \tilde{W}(\mathbf{x}_\tau) \leq 0$ at almost all $\tau < \bar{T}$;[8] thus, we have

$$W(\mathbf{x}_{\bar{T}}) \leq W(\mathbf{x}_0) + \int_0^{\bar{T}} \tilde{W}(\mathbf{x}_\tau) d\tau \leq W(\mathbf{x}_0).$$

Since $W(\mathbf{x}_0) < \bar{w}$ by $\mathbf{x}_0 \in X''$, we have $W(\mathbf{x}_{\bar{T}}) < \bar{w}$ in (10). This contradicts with $W(\mathbf{x}_{\bar{T}}) \geq \bar{w}$.

Therefore, the hypothesis (9) cannot hold: any Carathéodory solution trajectory $\{\mathbf{x}_t\}$ starting from X'' cannot escape X'' at any moment of time. That is, X'' is forward invariant. $\square$

Proof of Corollary 1

Proof. Define a Lyapunov function $W : \mathcal{X} \to \mathbb{R}$ and a decaying rate function $\tilde{W} : \mathcal{X} \to \mathbb{R}$ by

$$W(\mathbf{x}) := 2W_1(\mathbf{x}) + W_2(\mathbf{x}), \qquad \tilde{W}(\mathbf{x}) := 2\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x}) \qquad \text{for each } \mathbf{x} \in X_1.$$

Lipschitz continuity of W_1 and W_2 and lower semicontinuity of $\tilde{W}_1$ and $\tilde{W}_2$ are succeeded to those of W and $\tilde{W}$, respectively. It is immediate from assumptions a-i,iv), b-i,iv) and c) to see that

$$\begin{aligned} W(\mathbf{x}) &= 2W_1(\mathbf{x}) + W_2(\mathbf{x}) \geq 0, \\ \tilde{W}(\mathbf{x}) &= \tilde{W}_1(\mathbf{x}) + \{\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x})\} \leq 0, \\ DW(\mathbf{x})\dot{\mathbf{x}} &= 2DW_1(\mathbf{x})\dot{\mathbf{x}} + DW_2(\mathbf{x})\dot{\mathbf{x}} \leq 2\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x}) = \tilde{W}(\mathbf{x}) \end{aligned} \tag{11}$$

for any $\mathbf{x} \in X_1, \dot{\mathbf{x}} \in \boldsymbol{\mathcal{V}}(\mathbf{x})$ (for the last equation assuming that W_1 and W_2 are differentiable at $\mathbf{x}$).

Further, since $X^* \subset X_2$, it follows assumptions a-iii) and b-iii) that $W(\mathbf{x}) = \tilde{W}(\mathbf{x}) = 0$ if $\mathbf{x} \in X^*$; thus X^* is contained in $\mathrm{cl}\, X_1 \cap W^{-1}(0)$ and $\mathrm{cl}\, X_1 \cap \tilde{W}^{-1}(0)$ by $X^* \subset X_2 \subset X_1 \subset \mathrm{cl}\, X_1$. In contrary, assume $W(\mathbf{x}) = 0$ at $\mathbf{x} \in \mathrm{cl}\, X_1$ first. By assumptions a-i) and b-i), it must be the case that $W_1(\mathbf{x}) = 0$ and $W_2(\mathbf{x}) = 0$. The former implies $\mathbf{x} \in \mathrm{cl}\, X_2$ by assumption a-iii). Together with this, the latter implies $\mathbf{x} \in X^*$ by assumption b-iii). Separately from this, now assume $\tilde{W}(\mathbf{x}) = 0$ at $\mathbf{x} \in \mathrm{cl}\, X_1$. By assumptions a-ii) and c), it must be the case that $\tilde{W}_1(\mathbf{x}) = 0$ and $\tilde{W}_1(\mathbf{x}) + \tilde{W}_2(\mathbf{x}) = 0$.[9] The former implies $\mathbf{x} \in \mathrm{cl}\, X_2$ by assumption a-iii); besides, by plugging the former into the latter, we have $\tilde{W}_2(\mathbf{x}) = 0$. These two statements jointly imply $\mathbf{x} \in X^*$ by assumption b-iii). In sum, we

[8] A Carathéodory solution trajectory is differentiable at almost all moments of time, though it may not be so at all moments.

[9] Note that the latter condition alone cannot assure $\tilde{W}_2(\mathbf{x}) = 0$, since $\tilde{W}_2(\mathbf{x})$ could take a positive value unless $\mathbf{x}$ is in X_2.

have verified

$$\mathrm{cl}\, X_1 \cap W^{-1}(0) = \mathrm{cl}\, X_1 \cap \tilde{W}^{-1}(0) = X^*. \tag{12}$$

Note that the first equality is due to the fact that $X^* \subset X_1$ and thus $X^* \cap \mathrm{bd}\, X_1 = \emptyset$ since X_1 is open.

We have verified all the assumptions in Theorem 1; therefore, X^* is asymptotically stable. Notice that X_1 may not be forward invariant, but part i) of Theorem 1 assures that we can make some subset of X_1 as a basin of attraction to X^*. □

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