Intensity of Preferences in the Presence of Bivariate Risks

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Abstract: We show that ratios of cross partial derivatives of the utility measure the intensity of preferences towards bivariate risks in the expected utility model. In order to specify the meaning of these ratios, the definition of an $(n_1, n_2)^{th}$ degree increase in risk is combined with the concept of marginal rate of substitution between two increases in risks of different degrees. An intuitive explanation of these ratios and an illustration of their usefulness in understanding economic decisions are also provided.

JEL Classification: D81

Keywords: Ratio of cross partial derivatives; $(n_1, n_2)^{th}$ degree increase in bivariate risks; $(n_1, n_2)^{th}$ degree correlation aversion.

Introduction

Decision making usually occurs in the presence of multiple risks affecting several aspects of an individual's well-being. Indeed, risk management strategies (such as precautionary savings, the purchase of insurance contracts, preventive actions, or portfolio choices) are often adopted when agents are exposed to risks associated with attributes of the utility function other than wealth (health state, environment, relatives' wealth or health state,...). Papers dealing with precautionary saving in bivariate (Courbage and Rey, (2007)) or multivariate (Denuit, Eeckhoudt, and Menegatti, (2011); Jouini, Napp, and Nocetti, (2013); Courbage, (2014)) settings, with the effects of health risks on portfolio choices (Edwards, (2008); Crainich, Eeckhoudt, and Le Courtois, (2017)), or with investments improving future random health or environmental quality (Denuit, Eeckhoudt, and Menegatti, (2011); Jouini, Napp, and Nocetti, (2013)) offer illustrations of the way individuals behaving according to the expected utility model make that kind of decision. Note that non-financial decisions have also been treated in the literature: the effect of risky life expectancy on the choice of the intensity of a medical treatment in the presence of therapeutic risk has for instance been addressed in Bleichrodt, Crainich and Eeckhoudt (2011).

The above-mentioned papers can be classified into two categories: those interested in the *direction* of changes and those analyzing the *intensity* of changes in the decision variable once risks are either introduced or modified. Among the second category, Jouini, Napp, and Nocetti (2013) and Crainich, Eeckhoudt, and Le Courtois (2017) highlight that the trade-offs dictating agents' decisions depend on the value of various measures of the intensity of higher order multivariate risk attitude. In both papers, the latter are expressed as ratios wherein numerators are cross partial derivatives (of different orders) of the utility function and the denominator is the marginal utility of wealth. These measures share the same denominator because the cost of the economic decision is purely financial in both contributions. This cost is a forsaken current consumption that improves the future value of several attributes in Jouini, Napp, and Nocetti (2013) while it corresponds to reduced expected return when less risky assets are held in portfolios in Crainich, Eeckhoudt, and Le Courtois (2017).

When the effect of the introduction of multivariate risks on the strength of precautionary saving is analyzed, the denominators of the intensity measures of multivariate risk attitudes are second-order cross partial derivatives (Jouini, Napp, and Nocetti (2013); Courbage (2014)). However, as we illustrate below with a simple example, the cost of economic decisions can also be expressed in more general units, such as increases in higher order risks or increases in higher order correlations. In these cases, the denominators of the intensity measures take other forms. The purpose of our paper is to provide a comprehensive framework that encompasses all the measures of the intensity of preferences towards bivariate risks in the expected utility model.

Our work is based on three important contributions in risk theory: 1) the interpretation of the signs of successive cross partial derivatives of the utility function exposed in Eeckhoudt, Rey, and Schlesinger (2007); 2) the concept of n^{th} order increases in risk *à la* Ekern (1980); 3) the rate of substitution between two stochastic changes introduced in Liu and Meyer (2013). Based on these concepts, we propose an intuitive interpretation of the ratios of partial derivatives. As will be highlighted in the paper, these ratios will prove useful when dealing with risky decisions in bivariate settings.

To provide these interpretations, we first make use of the bivariate preference ordering introduced by Eeckhoudt, Rey, and Schlesinger (2007). The latter concept generalizes that of correlation aversion (Richard, (1975)), which refers to the preference for the dissociation between wealth and health[1] losses. Richard (1975) indicates that correlation aversion corresponds to $u^{(1,1)} < 0$ in the expected utility model[2]. Its generalization to higher orders is obtained when the dissociation principle is extended from losses to higher order zero-mean risks applied to both attributes. In the expected utility model, Eeckhoudt, Rey, and Schlesinger (2007) make the connection between this preference ordering and the signs of successive cross partial derivatives of the utility function *i.e.* the signs of $u^{(2,1)}$, $u^{(1,2)}$, $u^{(2,2)}$,..., $u^{(n_1,n_2)}$. Doing so, they establish the concept of $(n_1, n_2)^{th}$ degree correlation aversion.

We then apply the concept of n^{th} degree increase in risk (Ekern (1980)) to bivariate cases in order to obtain changes in bivariate distributions corresponding to the preference ordering defined in Eeckhoudt, Rey, and Schlesinger (2007). More precisely, while Ekern (1980) defines an n^{th} degree increase in risk by keeping the $n-1$ first moments of the distribution constant, Denuit et al. (2013) define an $(n_1, n_2)^{th}$ degree increase in bivariate risk as the movement from one distribution to another such that the $(n_1, n_2)^{th}$ comoment of these two distributions differs and such that three constraints are satisfied. Namely, the bivariate distributions must have: 1) the same first n_1 moments for the marginal distribution of the first variable; 2) the same first n_2 moments for the marginal distribution of the second variable; 3) the same first (n_1+n_2-1) comoments. It can then be shown that every $(n_1, n_2)^{th}$ degree correlation averse agent dislikes $(n_1, n_2)^{th}$ degree increases in bivariate risks (or, equivalently, $(n_1, n_2)^{th}$ degree increases in correlation).

Equipped with this definition, we extend Liu and Meyer (2013) and propose a general measure of the intensity of preferences towards bivariate risks. Liu and Meyer (2013) show that the existing measures ($\frac{-u''}{u'}$, $\frac{-u'''}{u''}$, $\frac{u'''}{u'}$,...) of the intensity of preferences towards risk in the univariate setting correspond to a rate of substitution between two increases in risk of different degrees. Specifically, suppose that $G(x)$ has more n^{th} degree risk than $F(x)$, that $H(x)$ has more m^{th} degree risk than $F(x)$ and that the agent is indifferent between $G(x)$ and $(1-T)F(x)+TH(x)$. Liu and Meyer (2013) indicate that T is proportional to the ratio $\frac{u^m}{u^n}$ that corresponds to a measure of the aversion towards increases in n^{th} degree risk. As noted by these authors, T is "*the weight that the decision maker is willing to put on $H(x)$ when forming a mixture of $F(x)$ and $H(x)$ to avoid having an n^{th} degree risk increase from $F(x)$ to $G(x)$ instead*". The same principle, applied to $(n_1, n_2)^{th}$ degree increases in correlation, is used in our paper in order to provide an intuitive interpretation of the measures of the intensity of bivariate risk preferences. Because the latter measures are expressed as ratios of cross partial derivatives, they do not depend on utility scales and allow us to compare individual behaviors towards bivariate risks.

The ratios of partial derivatives that we introduce generalize the existing measures of the intensity of bivariate preferences proposed by Jouini, Napp,

[1] Or any other attribute of the utility function.

[2] The following notation is adopted throughout the paper: $u^{(n_1,n_2)}$ refers to the n_1^{th} and n_2^{th} partial derivatives of the utility function with respect to its 1^{st} and 2^{nd} arguments, respectively.

and Nocetti (2013) and by Crainich, Eeckhoudt, and Le Courtois (2017). The latter authors exploit the concept of an $(n_1, n_2)^{th}$ degree correlation aversion (Eeckhoudt, Rey, and Schlesinger, (2007)) and extend the technique used by Crainich and Eeckhoudt (2008) who measure the intensity of downside risk aversion in an univariate setting. Namely, the intensity of preference towards a distribution $F(x)$ compared to a distribution $G(x)$ is equivalent to the maximum amount of money one is willing to sacrifice in order to be exposed to $F(x)$ rather than to $G(x)$. As a result, the existing measures of the intensity of preferences in bivariate or multivariate settings are all based on the marginal utility of wealth. We extend this interpretation by stating that the compensation might take other forms, such as changes in the marginal distributions corresponding to decreases in m^{th} degree risk or changes in the distribution corresponding to decreases in $(n_1, n_2)^{th}$ degree correlation.

Our paper is organized as follows. In Section 1, we introduce a saving decision that must be made in the presence of an environmental risk in order to illustrate the usefulness of a particular measure of the intensity of bivariate preferences, namely of $-\frac{u^{(1,2)}}{u^{(1,1)}}$. An explanation of the connection between our decision problem and that particular measure of preference towards bivariate risks is then established. Section 2 generalizes this illustration by adopting an approach based on the comoments of the distribution. Section 3 provides another generalization based on the concept of an increase in $(n_1, n_2)^{th}$ degree correlation. Section 4 concludes.

1 An illustration

Consider the following savings problem. An agent lives two periods during which he earns an identical income (w). Besides this income, the environment quality (e) she enjoys also enters in her utility function so that her preferences are represented by $u(w, e)$. We suppose that the utility of wealth and the utility of environmental quality are both increasing and concave, so that: $u^{(1,0)} > 0$, $u^{(0,1)} > 0$, $u^{(2,0)} < 0$, and $u^{(0,2)} < 0$. The agent has the opportunity to transfer money through savings from period 1 to period 2. To simplify this problem, suppose that the agent's utility function is the same across periods and that the interest rate and the rate of intertemporal preference are both null. Finally, no change in the environmental quality is anticipated between period 1 and period 2. The agent's maximization problem is given by:

$$\max_s \left(u(w - s, e) + u(w + s, e)\right). \tag{1}$$

It is straightforward to show that there are no savings $(s^* = 0)$ at the optimum. Suppose now that a new environmental program is implemented between period 1 and period 2. The expected improvement in environmental quality due to the program is positive, but the final effect of the program is risky. Suppose that this improvement is given by the random variable $\tilde{q}$ such that $\tilde{q} = q_0 + \tilde{\epsilon}$, with $q_0 > 0$, $E(\tilde{\epsilon}) = 0$, and where the variance of $\tilde{\epsilon}$ is denoted by $\sigma^2_{\tilde{\epsilon}}$. The agent's maximization then becomes:

$$\max_s \left(u(w - s, e) + Eu(w + s, e + \tilde{q})\right). \tag{2}$$

The first-order condition that is associated with this program and that defines the new optimal level of savings s^{**} is given by:

$$u^{(1,0)}(w - s^{**}, e) + Eu^{(1,0)}(w + s^{**}, e + \tilde{q}) = 0. \tag{3}$$

We approximate $Eu^{(1,0)}(w + s^{**}, e + \tilde{q})$ around $(w + s^{**}, e)$ with a limited Taylor expansion:

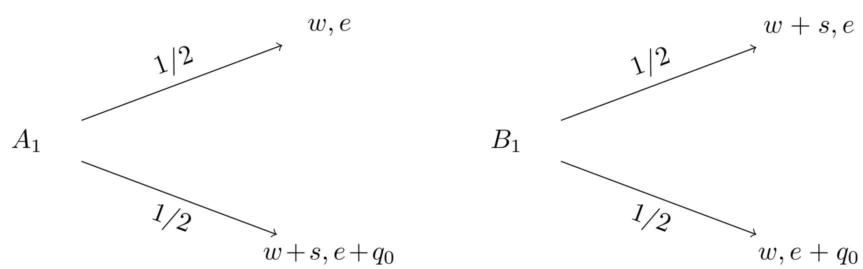
$$Eu^{(1,0)}(w+s^{**}, e+\tilde{q}) \simeq u^{(1,0)}(w+s^{**}, e)+q_0 u^{(1,1)}(w+s^{**}, e)+\frac{\sigma_{\tilde{\epsilon}}^2}{2}u^{(1,2)}(w+s^{**}, e).$$

When s^{**} is evaluated at $s^* = 0$, the first-order condition indicates the extent to which the implementation of the environmental program modifies savings. The agent saves $(s^{**} > s^* = 0)$ if:

$$-\frac{u^{(1,2)}(w, e)}{u^{(1,1)}(w, e)} > \frac{2q_0}{\sigma_{\tilde{\epsilon}}^2}.$$

Besides, the higher the gap between $\frac{\sigma_{\tilde{\epsilon}}^2}{q_0}$ and the ratio of cross partial derivatives of the bivariate utility function $-\frac{u^{(1,2)}(w,e)}{u^{(1,1)}(w,e)}$, the higher the difference between s^{**} and s^*. This result obtained in a very simplified model illustrates that the ratio of cross partial derivatives of the utility function explains decisions made by agents when they are exposed to bivariate risks.

In order to explain this result, we first use the interpretation of the concepts of correlation aversion and cross-prudence in health - or, equivalently, of the concepts of $(1, 1)^{th}$ and $(1, 2)^{th}$ degrees increase in correlation aversion - respectively defined by Richard (1975) and Eeckhoudt, Rey, and Schlesinger (2007). Consider the distributions A_1 and B_1 in the figure below and suppose that the agents are both $(1, 1)^{th}$ correlation averse $(u^{(1,1)} < 0)$ and $(1, 2)^{th}$ correlation averse $(u^{(1,2)} > 0)$.

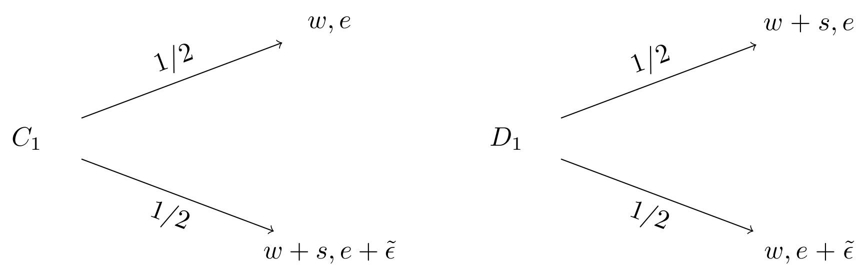


Following Richard (1975), it can be shown that $A_1 \prec B_1$ when individuals are $(1, 1)^{th}$ correlation averse. Indeed, using limited Taylor expansions, we obtain:

$$Eu(B_1) - Eu(A_1) \simeq -s\ q_0\ u^{(1,1)}(w, e).$$

This preference towards risk in a bivariate setting reflects the fact that agents prefer the sure gains (s on wealth and q_0 on the environmental attribute) to be dissociated (as in B_1) rather than associated in the same state of the world (as in A_1). In Richard (1975), the two outcome of the distributions A_1 and B_1 are two states of the world occurring with equal probability. Similar interpretations can be made if the outcomes are two time periods (assuming that there is no discount factor). Applied to our saving problem, $(1,1)^{th}$ correlation averse agents do not transfer of money from period 1 to period 2 in order not to concentrate the gains (the extra wealth due to savings and the expected improvement in environmental quality) in the same period.

Suppose now that same agents must choose between the distributions C_1 and D_1 described in the figure below.



Using Eeckhoudt, Rey, and Schlesinger (2007), we obtain $D_1 \prec C_1$ since individuals are $(1,2)^{th}$ degree correlation averse. Using limited Taylor expansions, it can be shown that:

$$Eu(C_1) - Eu(D_1) \simeq s\frac{\sigma^2_{\tilde{\epsilon}}}{2}u^{(1,2)}(w,e).$$

Agents who are $(1,2)^{th}$ degree correlation averse prefer to associate the improvement on the first attribute (due to s) with the pain (taking here the form of a zero mean risk $\tilde{\epsilon}$) on the second attribute of the utility function. As a result, and once again in order not to concentrate the gains in the same period, the risk associated with the environmental quality in period 2 provides an incentive to save for agents who are $(1,2)^{th}$ degree correlation averse.

Consequently, the implementation of the new environmental program has contradictory effects on correlation averse agents: the positive expected environmental value (q_0) deters savings as agents are $(1,1)^{th}$ degree correlation averse while the environmental risk ($\tilde{\epsilon}$) provides an incentive to save as they are $(1,2)^{th}$ degree correlation averse. The decisions to associate an extra unit of wealth to $e+q_0+\tilde{\epsilon}$ through savings rather than to e through dissavings depends on the relative values of $Eu(B_1) - Eu(A_1)$ and of $Eu(C_1) - Eu(D_1)$ *i.e.* on whether

$$-\frac{u^{(1,2)}(w,e)}{u^{(1,1)}(w,e)} > \frac{2q_0}{\sigma^2_{\tilde{\epsilon}}}.$$

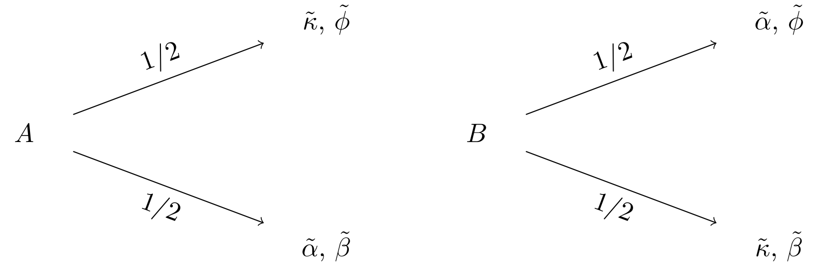
which corresponds to condition (1). The final effect of the environmental change on the saving decision thus depends not only on q_0 and $\tilde{\epsilon}$ but also on the relative aversions to $(1,1)^{th}$ degree and $(1,2)^{th}$ degree correlation. The left-hand side of the inequality sign of condition (1) actually corresponds to a marginal rate of substitution between the improvement in quality q_0 and the risk $\tilde{\epsilon}$ that is attached to it. As a result, $-\frac{u^{(1,2)}(w,e)}{u^{(1,1)}(w,e)}$ is a measure of the intensity of $(1,2)^{th}$ degree correlation aversion.

Note that other measures exist. For instance starting from distributions C_1 and D_1, Crainich, Eeckhoudt and Le Courtois (2017) suggest that the sure amount of money which, when subtracted from w in the upper branch of distribution C, restores indifference for $(1,2)^{th}$ degree correlation averse agents is one such measure. In that case, the intensity of $(1,2)^{th}$ degree correlation aversion is measured relatively to the marginal utility of wealth, *i.e.* relatively to $(1,0)^{th}$ degree correlation aversion. The next section generalizes this idea to describe the measures of preferences towards bivariate risks.

2 Substitutions of Lotteries

In the previous section, we illustrated the role played by the ratio of cross partial derivatives of a utility function in explaining a decision made by an agent exposed to a particular risky situation. This kind of trade-off can be extended to any risks. This is what we do in this section where an approach "in the small" is adopted.

To do so, consider first the lotteries A and B described below:



where $\tilde{\kappa}$, $\tilde{\phi}$, $\tilde{\alpha}$, and $\tilde{\beta}$ are four distinct random variables. The lotteries A and B respectively generalize the lotteries A_1 and B_1 described in the previous section if we assume that $\tilde{\kappa} = w$, $\tilde{\phi} = e$, $\tilde{\alpha} = w+s$ and $\tilde{\beta} = e+q_0$. Lottery B differs from lottery A in that the random variables $\tilde{\kappa}$ and $\tilde{\alpha}$ attached to the first argument of the utility function are swapped between the two possible states of the world. As we did in Section 1, we start by computing the difference in expected utility between the lotteries A and B.

Proposition 2.1. *We assume that all the partial derivatives of u are non-null. We have the following approximation:*

$$E(u(B)) - E(u(A)) \approx \frac{1}{2} \sum_{k=1}^{n_1} \sum_{h=1}^{n_2} \frac{E((\tilde{\alpha}^k - \tilde{\kappa}^k)(\tilde{\phi}^h - \tilde{\beta}^h))}{k!\ h!}\ u_{k,h}(0,0). \quad (4)$$

Proof. See Appendix. □

In the spirit of the utility premium of Friedman-Savage, a decomposition of the gain in expected utility resulting from being exposed to distribution B instead of distribution A is proposed in Eq. (4). Proposition 2.1 underlines that this gain is the weighted sum of differences between the moments and comoments of A and B, where the weights are the cross-derivatives of the utility function.

Note that the two distributions A and B are constructed by associating the same variables $\tilde{\kappa}$, $\tilde{\phi}$, $\tilde{\alpha}$, and $\tilde{\beta}$ in two different ways. A and B thus have the same marginal distributions and, consequently, the same moments. Besides, when $\tilde{\kappa}$ and $\tilde{\alpha}$ on the one hand, and $\tilde{\phi}$ and $\tilde{\beta}$ on the other, do not have the same higher-order moments ($\tilde{\alpha}$ and $\tilde{\kappa}$ do not have the same $(k)^{\text{th}}$ moment and $\tilde{\phi}$ and $\tilde{\beta}$ do not have the same $(h)^{\text{th}}$ moment), the comoments lower than $(k,h)^{\text{th}}$ of A and B are equal. Consequently, we deduce from Proposition 2.1 that the preference for B over A is only dictated by the sign of the $(k,h)^{\text{th}}$ cross-derivative of the utility function.

For instance, as the difference between distributions A_1 and B_1 (see Section 1) lies in the way the two sure gains attached to each argument of the utility function (s to the first argument and q_0 to the second one) are associated, the preference for A_1 or for B_1 depends on the sign of $u^{(1,1)}$. Likewise, the difference between distributions C_1 and D_1 is related to the way a sure gain attached to the first argument (s) and a risk attached to the second one ($\tilde{\epsilon}$) are associated: the preference for one of these two distribution over the other then depends on the sign of $u^{(1,2)}$.

In what follows, we assume that lottery A has more $(n_1,n_2)^{\text{th}}$ degree risk than lottery B in the sense defined below:

Definition 2.2 (More $(n_1,n_2)^{\text{th}}$ degree risk)**.** *We say that the lottery A has more $(n_1,n_2)^{th}$ degree risk than the lottery B if*[3]

$$\forall (k,h) < (n_1,n_2) \quad E((\tilde{\alpha}^k - \tilde{\kappa}^k)(\tilde{\phi}^h - \tilde{\beta}^h)) = 0 \tag{5}$$

and

$$(-1)^{n_1+n_2-1}\ E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2})) > 0. \tag{6}$$

The link with Ekern's (1980) definition of an n^{th} order increase in risk is straightforward: while the last comoment of A and B differs (see Eq. (6)), the moments of each of the two variables are equal and they share the same lower order comoments (see Eq. (5)).

We now come to the characterization of agents' preferences towards these two lotteries.

Definition 2.3 ($(n_1,n_2)^{\text{th}}$ degree risk aversion around 0)**.** *An agent u is $(n_1,n_2)^{th}$ degree risk averse around* 0 *if and only if* $(-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(0,0) > 0$.

This definition generalizes those of $(1,1)^{th}$ degree and of $(1,2)^{th}$ degree correlation aversion provided in Section 1. In what follows, being $(n_1,n_2)^{\text{th}}$ degree risk averse and being $(n_1,n_2)^{\text{th}}$ degree correlation averse are two terminologies referring to the same preference.

Proposition 2.4. *Consider an agent u who is $(n_1,n_2)^{th}$ degree risk averse around* 0*. This agent has the choice between two lotteries A and B, where the lottery A has more $(n_1,n_2)^{th}$ degree risk than the lottery B. Then, B is preferred to A and the difference in utilities between the two lotteries is given by*[4]

$$E(u(B)) - E(u(A)) \approx \frac{1}{2}\frac{E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2}))}{n_1!\ n_2!}\ u_{n_1,n_2}(0,0) > 0. \tag{9}$$

[3]The case $(k,h) < (n_1,n_2)$ includes the limit subcases $(n_1,h) < (n_1,n_2)$ with $h < n_2$, and $(k,n_2) < (n_1,n_2)$ with $k < n_1$.

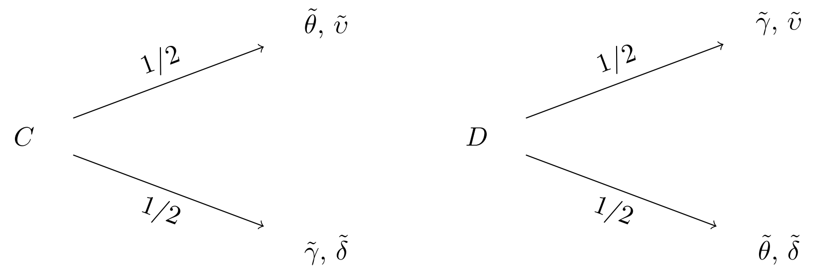
[4]If we assume in addition that $\tilde{\alpha}$ and $\tilde{\kappa}$ are independent from $\tilde{\phi}$ and $\tilde{\beta}$, we deduce that if

$$\forall (k,h) < (n_1,n_2) \quad E(\tilde{\alpha}^k) = E(\tilde{\kappa}^k) \text{ or } E(\tilde{\phi}^h) = E(\tilde{\beta}^h),$$

Proof. See Appendix. □

As in the lotteries used by Eeckhoudt, Rey and Schlesinger (2008), the distributions A and B are constructed so that they have equal lower order comoments: the preference that a $(n_1, n_2)^{\text{th}}$ degree risk averse individual has for the latter distribution over the former thus only results from the difference between their last comoment.

Let us now introduce the lotteries C and D:



where $\tilde{\theta}$, $\tilde{\delta}$, $\tilde{\gamma}$, and $\tilde{\upsilon}$ are independent random variables. The lotteries C and D respectively generalize the lotteries C_1 and D_1 described in Section 1 if we assume that $\tilde{\theta} = w$, $\tilde{\upsilon} = e$, $\tilde{\gamma} = w + s$ and $\tilde{\delta} = e + \tilde{\epsilon}$. Lottery D differs from lottery C in that the random variables $\tilde{\theta}$ and $\tilde{\gamma}$ attached to the first argument of the utility function are swapped between the two possible states of the world.

If we assume that the lottery C has more $(m_1, m_2)^{\text{th}}$ degree risk than the lottery D and that u is $(m_1, m_2)^{\text{th}}$ degree risk averse around 0, then we can use Proposition 2.4 to obtain:

$$E(u(D)) - E(u(C)) \approx \frac{1}{2}\frac{E((\tilde{\gamma}^{m_1} - \tilde{\theta}^{m_1})(\tilde{\upsilon}^{m_2} - \tilde{\delta}^{m_2}))}{m_1!\ m_2!}\ u_{m_1,m_2}(0,0) > 0. \quad (10)$$

As we did in the previous section, we can now define the marginal rate of substitution as the rate at which the agent is willing to exchange the transition from C to D against that from B to A.

Definition 2.5 (Substitution Rate)**.** *The marginal rate of substitution T_u is defined as follows:*

$$T_u = \frac{E(u(B)) - E(u(A))}{E(u(D)) - E(u(C))} \quad (11)$$

Since the lottery A has more $(n_1, n_2)^{\text{th}}$ degree risk than the lottery B and the lottery C has more $(m_1, m_2)^{\text{th}}$ degree risk than the lottery D, the substitution

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then

$$E(u(B)) - E(u(A)) \approx \frac{1}{2}\frac{E(\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})\ E(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2})}{n_1!\ n_2!}\ u_{n_1,n_2}(0,0). \quad (7)$$

and B is preferred to A provided that

$$E(\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})\ E(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2})\ u_{n_1,n_2}(0,0) > 0. \quad (8)$$

Further, if we assume that the sign of $u_{n_1,n_2}(0,0)$ is $(-1)^{n_1+n_2-1}$, then assuming the condition in (8) is equivalent to assuming that the sign of the product of moments $E(\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})\ E(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2})$ is also $(-1)^{n_1+n_2-1}$.

rate becomes

$$T_u = \frac{E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2}))}{E((\tilde{\gamma}^{m_1} - \tilde{\theta}^{m_1})(\tilde{\upsilon}^{m_2} - \tilde{\delta}^{m_2}))} \frac{m_1!\ m_2!}{n_1!\ n_2!} \frac{u_{n_1,n_2}(0,0)}{u_{m_1,m_2}(0,0)}, \tag{12}$$

where we use Eqs. (9) and (10) and where the ratio of moments in this equation is independent of the utility function u.

Suppose that a potential decision generates both an $(n_1, n_2)^{\text{th}}$ increase in bivariate risk (such as the transition from B to A) and an $(m_1, m_2)^{\text{th}}$ decrease in bivariate risk (such as the transition from C to D). Such a decision will be approved by agents as long as:

$$\begin{aligned} & E(u(D)) - E(u(C)) > E(u(B)) - E(u(A)) \Leftrightarrow \\ & \frac{u_{n_1,n_2}(0,0)}{u_{m_1,m_2}(0,0)} < \frac{E((\tilde{\gamma}^{m_1} - \tilde{\theta}^{m_1})(\tilde{\upsilon}^{m_2} - \tilde{\delta}^{m_2}))}{E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2}))} \frac{n_1!\ n_2!}{m_1!\ m_2!}. \end{aligned}$$

Finally, the marginal rate of substitution T allows the comparison between decisions made by two agents u and v, that is, it indicates the agent who is the more willing to trade an $(n_1, n_2)^{\text{th}}$ increase in bivariate risk against an $(m_1, m_2)^{\text{th}}$ decrease in bivariate risk:

Proposition 2.6. *Consider two decision makers u and v. When the lottery A has more $(n_1, n_2)^{th}$ degree risk than the lottery B and the lottery C has more $(m_1, m_2)^{th}$ degree risk than the lottery D, then,*

$$T_u \geq T_v \quad \Leftrightarrow \quad \frac{(-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(0,0)}{(-1)^{m_1+m_2-1}\ u^{(m_1,m_2)}(0,0)} \geq \frac{(-1)^{n_1+n_2-1}\ v^{(n_1,n_2)}(0,0)}{(-1)^{m_1+m_2-1}\ v^{(m_1,m_2)}(0,0)} \tag{13}$$

Proof. The proof is a direct consequence of Eq. (12) and of Eq. (6) applied to A and B, and to C and D, respectively. □

Proposition 2.6 establishes that the propensity to substitute changes in bivariate risks depends on the appropriate ratio of cross-derivatives of the utility function. It generalizes the problem exposed in Section 1 where savings was shown to simulateously increase $(1,1)^{th}$ degree risk and decrease $(1,2)^{th}$ degree risk. In that case, the individual whose $(1,1)^{th}$ degree correlation aversion relative to $(1,2)^{th}$ degree correlation aversion is lower (see condition (1)) has the higher propensity to save. Proposition 2.6 extends interpersonal comparisons to decisions leading to higher-order changes in bivariate risks. Namely, it states that individuals whose $(n_1, n_2)^{\text{th}}$ degree correlation aversion relative to $(m_1, m_2)^{\text{th}}$ degree correlation aversion is lower are more willing to accept $(n_1, n_2)^{\text{th}}$ degree increases in risk in exchange for $(m_1, m_2)^{\text{th}}$ degree decreases in risk.

3 Substitutions in the General Case

In the previous section, we adopted an approach "in the small" based on preferences that agents have for the comoments of bivariate distributions. We go one step further in the generalization process in this section by adopting an approach "in the large". The latter is based on the extension to the bivariate case of the definition of an "n^{th} degree increase in risk" introduced by Ekern (1980). Then, we adapt Liu and Meyer (2013) to the bivariate case in order to provide a general characterization of the intensity of bivariate preferences.

We consider a two dimensional random variable whose cumulative distribution function is denoted by F. We construct by successive integrations the function $F^{[k,h]}(.,.)$. Indeed, we have: $F^{[k,h]}(x,y) = \int_a^x F^{[k-1,h]}(s,y)ds$ and $F^{[k,h]}(x,y) = \int_a^y F^{[k,h-1]}(x,t)dt$, where the initial point is $F^{[1,1]}(x,y) = F(x,y)$.

Following Denuit et al. (2013), we define the concept of more high-order bivariate degree risks, ("$(n_1,n_2)^{\text{th}}$ degree increases in risk"'), which is an extension to the bivariate case of Ekern's definition of an "n^{th} degree increase in risk".

Definition 3.1 (More $(n_1,n_2)^{\text{th}}$ degree risk)**.** *A distribution G has more $(n_1,n_2)^{th}$ degree risk than a distribution F if and only if, for all $(k,h) \leq (n_1,n_2)$,*

$$G^{[k,h]}(s,b) = F^{[k,h]}(s,b) \quad \forall s \in [a,b], \tag{14}$$

and

$$G^{[k,h]}(b,t) = F^{[k,h]}(b,t) \quad \forall t \in [a,b], \tag{15}$$

and also

$$G^{(n_1,n_2)}(s,t) \geq F^{(n_1,n_2)}(s,t) \quad \forall (s,t) \in [a,b]^2. \tag{16}$$

Equipped with this definition, we extend Ekern (1980)'s result to the case of bivariate random vectors:

Proposition 3.2. *Let $\boldsymbol{X} = (X_1,X_2)$ and $\boldsymbol{Y} = (Y_1,Y_2)$ be bivariate random vectors that are respectively F-distributed and G-distributed. When G has more $(n_1,n_2)^{th}$ degree risk than F, then*

$$E(X_1^k\ X_2^h) = E(Y_1^k\ Y_2^h) \qquad \forall (k,h) < (n_1,n_2) \tag{17}$$

and

$$(-1)^{n_1+n_2}\ E(X_1^{n_1}\ X_2^{n_2}) \leq (-1)^{n_1+n_2}\ E(Y_1^{n_1}\ Y_2^{n_1}) \tag{18}$$

Proof. For brevity, we omit the proof, which is obvious and relies on multiple integrations by parts. □

This proposition states that the comparison of **X** and **Y** at order (n_1,n_2) is equivalent to the comparison of their non-centered comoment of order (n_1,n_2) when all of their lower-order non-centered moments and comoments are identical. These last constraints are expressed in (5) in the previous section and in (17) in this one. Proposition 3.2 makes the connection with the previous section where the distributions differ in their last comoments. However, note that there is no equivalence between (14), (15) and (16) on the one hand and (17) and (18) on the other: if any $(n_1,n_2)^{\text{th}}$ degree increase in risk implies that the bivariate distributions considered differ in their last comoment (their lower order comoments being equal), the opposite is not true. As a result, the approach based on the concept of $(n_1,n_2)^{\text{th}}$ degree increase in risk is more general than the comoments approach - based on approximations - that we developed in the previous section.

We now come to the characterization of agents faced with bivariate choices.

Definition 3.3 ($(n_1,n_2)^{\text{th}}$ degree risk aversion)**.** *An agent is $(n_1,n_2)^{th}$ degree risk averse if and only if*

$$(-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(s,t) > 0 \quad \forall (s,t) \in [a,b]^2.$$

Then, we can relate the comparisons of $(n_1,n_2)^{\text{th}}$ degree risks to the preferences of agents as follows:

Proposition 3.4. *G has more $(n_1,n_2)^{th}$ degree risk than F if and only if every $(n_1,n_2)^{th}$ degree risk averter prefers F to G.*

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be bivariate random vectors that are F-distributed and G-distributed, respectively. The proof, which is obvious, relies on the computation of

$$E(u(\mathbf{X})) - E(u(\mathbf{Y})) = \int_a^b \int_a^b u(s,t)\ dF(s,t) - \int_a^b \int_a^b u(s,t)\ dG(s,t)$$

via multiple integrations by parts, and on the study of its sign.

□

Proposition 3.4 in this section is the equivalent of Proposition 2.4 in the previous section. Both of these propositions refer to preferences that $(n_1,n_2)^{\text{th}}$ degree risk averse agents have towards bivariate distributions. But they say nothing about the intensity of these preferences or about the choices that two different $(n_1,n_2)^{\text{th}}$ degree risk averse agents might make. The transition from the concept of a "direction of preferences" to that of an "intensity of preferences" in bivariate settings is established - as in Sections 1 and 2 - through the comparison between two changes in distributions.

To do so, we extend to the bivariate case a technique introduced by Liu and Meyer (2013) in the univariate context. Assume that $G(x,y)$ has more $(n_1,n_2)^{\text{th}}$ degree risk than $F(x,y)$ and that $H(x,y)$ has more $(m_1,m_2)^{\text{th}}$ degree risk than $F(x,y)$. What is the value of T such that an agent is indifferent between $G(x,y)$ and $(1-T)F(x,y) + TH(x,y)$? The higher T, the more the agent is sensitive to $(n_1,n_2)^{\text{th}}$ degree increases in risk relative to $(m_1,m_2)^{\text{th}}$ degree increases in risk. As a result, T indicates the intensity of preferences in bivariate settings.

It is straightforward to show that for the agent u, the value of T - denoted T_u - is given by:

Definition 3.5.

$$T_u = \frac{\int\limits_a^b \int\limits_a^b u(s,t)(dF(s,t) - dG(s,t))}{\int\limits_a^b \int\limits_a^b u(s,t)(dF(s,t) - dH(s,t))}$$

As in Liu and Meyer (2013), the proposed measure of intensity of bivariate risk aversion is a ratio of two expected utility changes. While the Arrow-Pratt measure of risk aversion is sufficient to deal with the introduction of risks or with changes in risk "in the small", Ross (1981) has shown that a stronger measure of the increase in risk aversion was required once increases in risk or once changes in risk "in the large" were addressed. We now indicate how the generalization of the Ross measure of "stronger risk aversion" is helpful when higher-order increases in bivariate risk "in the large" are considered. The Arrow-Pratt and Ross increases in higher-order bivariate risk aversion are respectively defined as follows:

Definition 3.6. *u is $((n_1, n_2)/(m_1, m_2))^{th}$ degree more risk averse than v if, for all $(s, t) \in [a, b]^2$,*

$$\frac{(-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(s,t)}{(-1)^{m_1+m_2-1}\ u^{(m_1,m_2)}(s,t)} \geq \frac{(-1)^{n_1+n_2-1}\ v^{(n_1,n_2)}(s,t)}{(-1)^{m_1+m_2-1}\ v^{(m_1,m_2)}(s,t)},$$

whereas u is $((n_1, n_2)/(m_1, m_2))^{th}$ degree Ross more risk averse than v if, for all $(s, t) \in [a, b]^2$ and for all $(w, z) \in [a, b]^2$,

$$\frac{(-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(s,t)}{(-1)^{m_1+m_2-1}\ u^{(m_1,m_2)}(w,z)} \geq \frac{(-1)^{n_1+n_2-1}\ v^{(n_1,n_2)}(s,t)}{(-1)^{m_1+m_2-1}\ v^{(m_1,m_2)}(w,z)}.$$

These definitions allow the interpersonal comparison of the propensity to exchange an $(n_1, n_2)^{\text{th}}$ increase in bivariate risk against an $(m_1, m_2)^{\text{th}}$ decrease in bivariate risk. Combining what precedes, the following Proposition can indeed be established:

Proposition 3.7. *We consider two agents u and v that are each both $(n_1, n_2)^{th}$ degree risk averse and $(m_1, m_2)^{th}$ degree risk averse. The following statements are equivalent:*

(i) u is $((n_1, n_2)/(m_1, m_2))^{th}$ degree Ross more risk averse than v on $[a, b]^2$, so that there exists $\lambda > 0$ such that $\frac{u^{(n_1,n_2)}(s,t)}{v^{(n_1,n_2)}(s,t)} \geq \lambda \geq \frac{u^{(m_1,m_2)}(w,z)}{v^{(m_1,m_2)}(w,z)}$ for all $(s, t) \in [a, b]^2$ and $(w, z) \in [a, b]^2$.

(ii) There exist $\lambda > 0$ and $\phi : [a, b]^2 \to \mathbb{R}$ such that $u = \lambda v + \phi$ and such that $(-1)^{m_1+m_2-1}\phi^{(m_1,m_2)}(s, t) \leq 0$ and $(-1)^{n_1+n_2-1}\phi^{(n_1,n_2)}(s, t) \geq 0$ for all $(s, t) \in [a, b]^2$.

(iii) $T_u \geq T_v$ for all F, G, and H such that G has more $(n_1, n_2)^{th}$ degree risk than F and H has more $(m_1, m_2)^{th}$ degree risk than F

Proof. See Appendix.

□

The willingness to trade $(n_1, n_2)^{\text{th}}$ increases in risk against $(m_1, m_2)^{\text{th}}$ decreases in risk depends on the intensity T of preferences in bivariate settings. Part (i) of Proposition 3.7 establishes the condition under which an individual whose preferences are represented by the utility function u is more Ross bivariate risk averse than another individual whose preferences are represented by v. Parts (ii) of Proposition 3.7 defines the transformation of the bivariate utility function that preserves this partial order. Part (iii) of the Proposition underlines that this ordering determines the propensity to substitute two kinds of change in bivariate risks and, as a result, is a measure of the intensity of preferences towards risk in the bivariate setting.

In Section 2, we adopted an approach "in the small" based on the comparison between comoments of the distribution. In such a case, one can compare the decisions made by two agents by using the Arrow-Pratt concept of an $((n_1, n_2)/(m_1, m_2))^{\text{th}}$ increase in bivariate risk aversion (see Proposition 2.6). Because an approach "in the large" is adopted in this section, the Proposition indicates that the notion of $((n_1, n_2)/(m_1, m_2))^{\text{th}}$ increase in bivariate risk aversion in the Ross sense is required in order to compare the preferences expressed by two agents agents exposed to changes in bivariate risk.

4 Conclusion

Decision-makers consider various aspects of their well-being (wealth, health, environment quality,...) when they make choices that modify the risks they are exposed to. In this paper, we show how the ratios of cross partial derivatives of the utility function measure the intensity of preferences that individuals display in bivariate settings. When an approach "in the small" based on the comparison of the comoments of the distributions is adopted, we show that the decisions agents make depend on their $((n_1, n_2)/(m_1, m_2))^{\text{th}}$ Arrow-Pratt risk aversion. Then, we extend Liu and Meyer (2013) to the bivariate case to show that the propensity to substitute $(n_1, n_2)^{\text{th}}$ increases in risk for $(m_1, m_2)^{\text{th}}$ decreases in risk "in the large" can be explained by the concept of an "increase in $((n_1, n_2)/(m_1, m_2))^{\text{th}}$ Ross risk aversion". These last ratios constitute general measures of the intensity of preferences towards risk in bivariate settings.

Appendix

Proof of Proposition 2.1

From the definition of the lotteries, we have:

$$\Delta \overset{def}{=} E(u(B)) - E(u(A)) = \frac{1}{2}E(u(\tilde{\alpha}, \tilde{\phi})) + \frac{1}{2}E(u(\tilde{\kappa}, \tilde{\beta})) - \frac{1}{2}E(u(\tilde{\kappa}, \tilde{\phi})) - \frac{1}{2}E(u(\tilde{\alpha}, \tilde{\beta}))$$

or

$$\Delta = \frac{1}{2}\left[E(u(\tilde{\alpha}, \tilde{\phi})) - E(u(\tilde{\kappa}, \tilde{\phi}))\right] - \frac{1}{2}\left[E(u(\tilde{\alpha}, \tilde{\beta})) - E(u(\tilde{\kappa}, \tilde{\beta}))\right].$$

Then,

$$\Delta \approx \frac{1}{2}E\left[\sum_{k=1}^{n_1} \frac{\tilde{\alpha}^k}{k!}\ u_{k,0}(0, \tilde{\phi}) - \sum_{k=1}^{n_1} \frac{\tilde{\kappa}^k}{k!}\ u_{k,0}(0, \tilde{\phi})\right] - \frac{1}{2}E\left[\sum_{k=1}^{n_1} \frac{\tilde{\alpha}^k}{k!}\ u_{k,0}(0, \tilde{\beta}) - \sum_{k=1}^{n_1} \frac{\tilde{\kappa}^k}{k!}\ u_{k,0}(0, \tilde{\beta})\right],$$

so that

$$\Delta \approx \frac{1}{2}E\left[\sum_{k=1}^{n_1} \frac{\tilde{\alpha}^k - \tilde{\kappa}^k}{k!}\ \left(u_{k,0}(0, \tilde{\phi}) - u_{k,0}(0, \tilde{\beta})\right)\right].$$

Developing with respect to the second argument, we obtain:

$$\Delta \approx \frac{1}{2}E\left[\sum_{k=1}^{n_1}\sum_{h=1}^{n_2} \frac{\tilde{\alpha}^k - \tilde{\kappa}^k}{k!}\ \frac{\tilde{\phi}^h - \tilde{\beta}^h}{h!}\ u_{k,h}(0,0)\right].$$

which is our result.

Proof of Proposition 2.4

We first cancel the terms in Eq. (4) that are null in Eq. (5). This operation allows us to write:

$$E(u(B)) - E(u(A)) \approx \frac{1}{2} \frac{E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2}))}{n_1!\ n_2!}\ u_{n_1,n_2}(0,0),$$

which is equivalent to

$$E(u(B)) - E(u(A)) \tag{19}$$

$$\approx \frac{1}{2}\left((-1)^{n_1+n_2-1}\frac{E((\tilde{\alpha}^{n_1} - \tilde{\kappa}^{n_1})(\tilde{\phi}^{n_2} - \tilde{\beta}^{n_2}))}{n_1!\ n_2!}\right)\ \left((-1)^{n_1+n_2-1}u_{n_1,n_2}(0,0)\right). \tag{20}$$

Then, we observe that the first bracket is positive thanks to Eq. (6) and that the second bracket is positive thanks to the $(n_1, n_2)^{\text{th}}$ degree risk aversion around 0 property of u.

Proof of Proposition 3.7

The equivalences are shown by generalizing the arguments in Liu and Meyer (2013).

$(i) \Rightarrow (ii)$. Using λ defined in (i), we construct ϕ as follows:

$$\phi(s,t) = u(s,t) - \lambda\ v(s,t) \qquad \forall (s,t) \in [a,b]^2.$$

We readily check that

$$\begin{aligned}&(-1)^{m_1+m_2-1}\phi^{(m_1,m_2)}(s,t)\\ &\quad = (-1)^{m_1+m_2-1}u^{(m_1,m_2)}(s,t) - \lambda\ (-1)^{m_1+m_2-1}v^{(m_1,m_2)}(s,t) \leq 0\end{aligned}$$

and

$$\begin{aligned}&(-1)^{n_1+n_2-1}\phi^{(n_1,n_2)}(s,t)\\ &\quad = (-1)^{n_1+n_2-1}u^{(n_1,n_2)}(s,t) - \lambda\ (-1)^{n_1+n_2-1}v^{(n_1,n_2)}(s,t) \geq 0.\end{aligned}$$

$(ii) \Rightarrow (iii)$. From the definition of T_u, we have:

$$\int_a^b \int_a^b u(s,t)\ \ [(1-T_u)dF(s,t) + T_u dH(s,t)] = \int_a^b \int_a^b u(s,t)\ dG(s,t).$$

Then,

$$\int_a^b \int_a^b u(s,t)\ dG(s,t) = \int_a^b \int_a^b (\lambda v(s,t) + \phi(s,t))\ dG(s,t)$$

and

$$\int_a^b \int_a^b (\lambda v(s,t) + \phi(s,t))\ dG(s,t) \leq \int_a^b \int_a^b \lambda v(s,t)\ dG(s,t) + \int_a^b \int_a^b \phi(s,t)\ dF(s,t)$$

because ϕ shows $(n_1, n_2)^{\text{th}}$ degree risk aversion. Further, using the definition of T_v and the fact that ϕ shows $(m_1, m_2)^{\text{th}}$ degree risk taking, we have:

$$\int_a^b \int_a^b \lambda v(s,t)\ dG(s,t) + \int_a^b \int_a^b \phi(s,t)\ dF(s,t) \leq$$
$$\int_a^b \int_a^b \lambda v(s,t)\ \left[(1-T_v)dF(s,t) + T_v dH(s,t)\right] + \int_a^b \int_a^b \phi(s,t)\ \left[(1-T_v)dF(s,t) + T_v dH(s,t)\right],$$

so that

$$\int_a^b \int_a^b \lambda v(s,t)\ dG(s,t) + \int_a^b \int_a^b \phi(s,t)\ dF(s,t) \leq \int_a^b \int_a^b u(s,t)\ \left[(1-T_v)dF(s,t) + T_v dH(s,t)\right].$$

Recombining the above equalities and inequalities shows that $T_u \geq T_v$.

$(iii) \Rightarrow (i)$. By integration by parts, we can show that

$$\int_a^b \int_a^b (-1)^{k+h-1}\ u^{(k,h)}(s,t)\ (G^{[k,h]}(s,t) - F^{[k,h]}(s,t))\ ds\ dt$$
$$\int_a^b \left[(-1)^{k+h-1}\ u^{(k,h)}(s,t)\ (G^{[k+1,h]}(s,t) - F^{[k+1,h]}(s,t))\right]_a^b dt$$
$$- \int_a^b \int_a^b (-1)^{k+h-1}\ u^{(k+1,h)}(s,t)\ (G^{[k+1,h]}(s,t) - F^{[k+1,h]}(s,t))\ ds\ dt.$$

Using conditions (14) and (15), we obtain:

$$\int_a^b \int_a^b (-1)^{k+h-1}\ u^{(k,h)}(s,t)\ (G^{[k,h]}(s,t) - F^{[k,h]}(s,t))\ ds\ dt =$$
$$\int_a^b \int_a^b (-1)^{(k+1)+h-1}\ u^{(k+1,h)}(s,t)\ (G^{[k+1,h]}(s,t) - F^{[k+1,h]}(s,t))\ ds\ dt,$$

which shows that an identical formula prevails at orders k and $k+1$ on wealth. By extension, we have:

$$\int_a^b \int_a^b\ u(s,t)\ (dF(s,t) - dG(s,t))\ ds\ dt =$$
$$\int_a^b \int_a^b (-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(s,t)\ (G^{[n_1,n_2]}(s,t) - F^{[n_1,n_2]}(s,t))\ ds\ dt.$$

Therefore,

$$T_u \geq T_v \quad \Leftrightarrow$$

$$\frac{\int_a^b\int_a^b (-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(s,t)\ (G^{[n_1,n_2]}(s,t) - F^{[n_1,n_2]}(s,t))\ ds\ dt}{\int_a^b\int_a^b (-1)^{m_1+m_2-1}\ u^{(m_1,m_2)}(s,t)\ (H^{[m_1,m_2]}(s,t) - F^{[m_1,m_2]}(s,t))\ ds\ dt} \geq$$

$$\frac{\int_a^b\int_a^b (-1)^{n_1+n_2-1}\ v^{(n_1,n_2)}(s,t)\ (G^{[n_1,n_2]}(s,t) - F^{[n_1,n_2]}(s,t))\ ds\ dt}{\int_a^b\int_a^b (-1)^{m_1+m_2-1}\ v^{(m_1,m_2)}(s,t)\ (H^{[m_1,m_2]}(s,t) - F^{[m_1,m_2]}(s,t))\ ds\ dt}$$

Next, we assume that (i) does not hold: we can find two compact sets $[c,d]^2$ and $[e,f]^2$ and $\mu > 0$ such that

$$\frac{u^{(n_1,n_2)}(s,t)}{v^{(n_1,n_2)}(s,t)} < \mu < \frac{u^{(m_1,m_2)}(w,z)}{v^{(m_1,m_2)}(w,z)}$$

for all $(s,t) \in [c,d]^2$ and $(w,z) \in [e,f]^2$.

Because we assume that v is $(n_1,n_2)^{\text{th}}$ and $(m_1,m_2)^{\text{th}}$ degree risk averse, we have:

$$(-1)^{n_1+n_2-1} u^{(n_1,n_2)}(s,t) < \mu\ (-1)^{n_1+n_2-1} v^{(n_1,n_2)}(s,t)$$

for all $(s,t) \in [c,d]^2$ and

$$(-1)^{m_1+m_2-1} u^{(m_1,m_2)}(w,z) > \mu\ (-1)^{m_1+m_2-1} v^{(m_1,m_2)}(w,z)$$

for all $(w,z) \in [e,f]^2$

Then, choosing $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$ such that $\tilde{G} - \tilde{F} > 0$ on $[c,d]^2$, $\tilde{H} - \tilde{F} > 0$ on $[e,f]^2$ and such that these two differences are null outside the compact sets, we can write:

$$\int_a^b\int_a^b (-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(s,t)\ (\tilde{G}^{[n_1,n_2]}(s,t) - \tilde{F}^{[n_1,n_2]}(s,t))\ ds\ dt \quad <$$

$$\mu \int_a^b\int_a^b (-1)^{n_1+n_2-1}\ v^{(n_1,n_2)}(s,t)\ (\tilde{G}^{[n_1,n_2]}(s,t) - \tilde{F}^{[n_1,n_2]}(s,t))\ ds\ dt$$

and

$$\int_a^b\int_a^b (-1)^{m_1+m_2-1}\ u^{(m_1,m_2)}(s,t)\ (\tilde{H}^{[m_1,m_2]}(s,t) - \tilde{F}^{[m_1,m_2]}(s,t))\ ds\ dt \quad >$$

$$\mu \int_a^b\int_a^b (-1)^{m_1+m_2-1}\ v^{(m_1,m_2)}(s,t)\ (\tilde{H}^{[m_1,m_2]}(s,t) - \tilde{F}^{[m_1,m_2]}(s,t))\ ds\ dt$$

so that

$$\frac{\int_a^b \int_a^b (-1)^{n_1+n_2-1}\ u^{(n_1,n_2)}(s,t)\ (\tilde{G}^{[n_1,n_2]}(s,t) - \tilde{F}^{[n_1,n_2]}(s,t))\ ds\ dt}{\int_a^b \int_a^b (-1)^{m_1+m_2-1}\ u^{(m_1,m_2)}(s,t)\ (\tilde{H}^{[m_1,m_2]}(s,t) - \tilde{F}^{[m_1,m_2]}(s,t))\ ds\ dt} < \frac{\int_a^b \int_a^b (-1)^{n_1+n_2-1}\ v^{(n_1,n_2)}(s,t)\ (\tilde{G}^{[n_1,n_2]}(s,t) - \tilde{F}^{[n_1,n_2]}(s,t))\ ds\ dt}{\int_a^b \int_a^b (-1)^{m_1+m_2-1}\ v^{(m_1,m_2)}(s,t)\ (\tilde{H}^{[m_1,m_2]}(s,t) - \tilde{F}^{[m_1,m_2]}(s,t))\ ds\ dt}$$

which is a contradiction. Therefore (i) holds.

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