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On the asymptotic normality of the R-estimators of the slope parameters of simple linear regression models with positively dependent errors

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Abstract

The purpose of this paper is to prove the asymptotic normality of the rank estimator of the slope parameter of a simple linear regression model with stationary associated errors. This result follows from a uniform linearity property for a linear rank statistics that we establish under general conditions on the dependence of the errors. We prove also a tightness criterion for weighted empirical process constructed from associated triangular arrays. This criterion is needed for the proofs which are based on that of Koul (1977) and of Louhichi (2000).

1 Introduction

Time series regression models constitute a rich class of statistical models used in several fields such as in finance, in econometrics, in biology or for environmental studies. A special interest is dedicated to simple linear regression models with correlated errors :

$$Z_i = \alpha + \beta x_i + \epsilon_i, \;\; 1 \le i \le n, \tag{1}$$

where the $(x_i)_{1\le i\le n}$ are known regression constants not all equal, α and β stand, respectively, for the intercept and the slope parameters, and the errors $(\epsilon_i)_{1\le i\le n}$ constitute a sequence of strictly stationary random variables with a marginal distribution function F and absolutely continuous density f.

The method of least squares is a standard approach often used to estimate the parameters α and β in the linear model (1). It's known, from the Gauss-Markov theorem, that the least square estimators of those parameters, for uncorrelated errors with zero expectation and equal variances, have the nice property of being the best linear unbiased estimators. Those estimators use, however, all the values of the observations and thus they are vulnerable to gross errors. *Estimates based on appropriate rank*

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statistics have excellent robustness prospects and they are, for independent and identically distributed errors, distribution free in a true sense (we refer to Chapter 4, page 363 of the the book of Sidak et al. (1999)).

Estimators derived from rank statistics (in short, R-estimators) were introduced by Hodges and Lehmann in 1963. It was for location (or center of symmetry) of symmetric distribution. But a full-scale development of rank based statistical methods seem to have sparked in 1945 with Wilcoxon rank-sum test in Wilcoxon (1945) and by Kendall in 1948. It was a test statistics for two sample problem. The asymptotic theory for R-estimators had been studied with other types of estimators such as M-estimators (maximum likelihood type estimators) and L-estimators (linear combination of order statistics) during 1960's to 1980's, as can be seen in the books such as P. J. Huber and E. M. Ronchetti (2009), J. Jurecková and P. K. Sen (1996) and E. L. Lehmann (1975) and (1983). There, an accuracy of estimators, smallness of asymptotic variance or asymptotic efficiency, are the central issue in the location problem. Hodge-Lehmann estimator that is a representative R-estimator shows up as a counterparty against sample means that is a representative maximum likelihood M-type estimators as well as a representative L-estimator, where Hodge-Lehmann estimators proved a better robustness in a certain neighborhood of normal distribution than a sample mean unless it is modified well like a trimmed mean. These theoretical results in location problem are transferred, under certain conditions, to the efficiency issues for a linear regression models that is a straight extension of a location model.

In this paper our main interest is in robust estimates and in estimations based on ranks. We focus on the R-estimation of the parameter β of the model (1). This estimator is constructed from suitable linear rank statistics: for every $\Delta \in \mathbb{R}$, and each $i, 1 \le i \le n$, let

$R_{i\Delta} =$ Rank of $(Z_i - \Delta x_i)$ among $(Z_j - \Delta x_j)_{1 \le j \le n}$ provided all components $(Z_i - \Delta x_i)_{1 \le i \le n}$ are different.

Let φ be a nondecreasing and right continuous function defined on $[0, 1]$ (a score function). Let us consider the following linear rank statistics, originally derived from rank tests of the hypothesis specifying the value of the location parameter,

$$S_x(\Delta) := \sum_{i=1}^{n} (x_i - \bar{x}_n) \varphi(\frac{R_{i\Delta}}{n+1}), \tag{2}$$

where $\bar{x}_n = n^{-1} \sum_{i=1}^{n} x_i$. The statistic $S_x(\Delta)$ is almost surely a monotone[1] (nonincreasing) step function of Δ (cf. Theorem 2.1 in Jurecková (1969)). This statistics is not defined in the points of the finite set:

$$\{\Delta, Z_i - \Delta x_i = Z_j - \Delta x_j \text{ for at least one pair } (i, j)\}.$$

As it was suggested in Jurecková (1969), the definition of $S_x(\Delta)$ may be complete at its discontinuity points as to be continuous either from the left or from the right. However, if the underling distribution F is continuous, then this occurs only in a set of probability zero for any given values of $(x_i)_{1 \le i \le n}$. We shall then suppose in the sequel that $S_x(\Delta)$ is well defined for any real Δ. The R-estimator of β, $\hat{\beta}$, based on the observation of $(x_i, Z_i)_{1 \le i \le n}$, is any value of Δ for which $S_x(\Delta)$ is as near to 0 as

[1]Monotonicity of $S_x(\Delta)$ as a function of Δ is discussed without using any probability argument: without iid assumptions and also without an underlying distribution F.

possible i.e.

$$\hat{\beta} = \text{argmin}_{\Delta>0}|S_x(\Delta)|, \tag{3}$$

(let us note that $\hat{\beta}$ can also be defined as any value of Δ that is both, greater than $\sup\{\Delta, S_x(\Delta) > 0\}$ and less than $\inf\{\Delta, S_x(\Delta) < 0\}$). Such estimator may not be uniquely determined. An other basic property of the statistic $S_x(\Delta)$ that plays the most fundamental role in the asymptotic theory of R-estimation of regression parameters is the asymptotic uniform linearity : for independent and identically distributed errors $(\epsilon_i)_{1\leq i\leq n}$ with a finite Fisher information,

$$I(f) = \int_{-\infty}^{\infty} \frac{f'^2(x)}{f(x)} dx < \infty, \tag{4}$$

and under some conditions on the regression constants $(x_i)_{1\leq i\leq n}$ (cf. Theorem 3.1 in Jurecková (1969)), the asymptotic uniform linearity property states that,

$$\lim_{n\to\infty} P\left(\sup_{|\Delta|\leq C} |S_x(\Delta) - S_x(0) - \Delta b_n(\varphi)| \geq \epsilon\sqrt{\text{Var}(S_x(\Delta))}\right) = 0, \tag{5}$$

for any $\epsilon > 0, C > 0$ where $b_n(\varphi) = -\sum_{i=1}^n (x_i - \bar{x}_n)^2 \int_0^1 \varphi(u)\varphi(u,f)du$ with

$$\varphi(u,f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}, \text{ where } F^{-1}(u) = \inf\{x \in \mathbb{R}, F(x) \geq u\}. \tag{6}$$

The uniform asymptotic linearity property is the basic tool for the proof of the asymptotic normality of R-estimators. Those results were proved when errors $(\epsilon_i)_{1\leq i\leq n}$ are independent and identically distributed (see for instance Koul (1969, 1970), Jurecková (1969)) or strongly mixing (Koul (1977)). R-estimators of the regression parameters based on general linear rank statistics was iniciated by Adichie (1967) and developed by Jurecková (1969, 1971) and Sen (1969) and Koul (1971). R-estimators for a simple linear regression model is very well explained by Jaeckel (1972). We refer also to the books by Puri and Sen (1985) or to Jureckovà and Sen (1996). In the context of time series analysis, R-estimation has been also developed (Allal (1991), Hallin et al. (1985), Hallin and Puri (1988), Koul and Saleh (1993), Allal et al. (2001)).

The asymptotic variances of R-estimators depend on the score function φ and the underlying common distribution function F of the errors (the details can be seen in the above references). This means that the choice of the score function (against the underlying distribution) determines the accuracy of the estimator. In the case that the functional form F of the underlying distribution is known to the statistician, she/he will choose the best score function against it in order to obtain the possible minimum asymptotic variance which is expressed with Fisher information amount. However, it is usual that such a functional form is not known to them. Then, we need to evaluate how well their choice of score functions perform, i.e. how accurate their statistical inference would be against the unknown underlying distribution (of real data) that is the discussion in robustness of estimators. For example if they know the underlying distribution is Normal, then they would construct the score function φ as the inverse function of distribution function of standard normal Φ^{-1}, called Normal score (note that this is unbounded), if it is Logistic, then we would take $\varphi(t) = 2t-1$ (note that this is bounded) (see page 69 in Huber and Ronchetti (2009)). The arguments on the choice of score functions in relations to asymptotic accuracy transfer well to the estimation problems in linear regression models. As seen in the references for details, the score functions φ determine the functional forms of the rank statistics and, therefore, what functions of data the R-estimators are. In the references, Lehmann (1983): the score function is $K(\cdot)$ in page 383, $J(\cdot)$ in Huber and Ronchetti (2009) page 61, then, Φ in Jurecková and Sen (1996) page 106 and 236.

The purpose of this paper is to investigate the behavior of the R-estimator of β for the model (1) when the errors $(\epsilon_i)_{1\le i\le n}$ are a stationary sequence of associated random variables : for any $n \geq 1$, and any bounded and nondecreasing functions h and k,

$$\mathrm{Cov}(h(\epsilon_1, \cdots, \epsilon_n), k(\epsilon_1, \cdots, \epsilon_n)) \geq 0.$$

An autoregressive sequence with positive slope parameter, is a typical example of associated sequence. In practice the model can be used, for example, in financial industry how much a return of a portfolio (say hedge fund or mutual fund portfolio) be explained by market representing indices such as stock and/or bond indices, where the residuals often show weak dependence.

Our main result is Theorem 1, stated in Section 1 below. It proves the asymptotic normality of the R-estimator for associated errors and for bounded score functions φ. In section 2 we give, in Theorem 2, conditions under which the asymptotic normality of the R-estimator holds for any stationary dependent errors. Theorem 2 is a consequence of the uniform asymptotic linearity of $S_x(\Delta)$ stated in Proposition 1 for stationary dependent errors. Proposition 2 and Proposition 3 give criteria for the tightness property for weighted empirical processes constructed from associated sequences: they allow to check the conditions of Proposition 1 in the case of associated errors and then to complete the proof of Theorem 1 (as a consequence of Theorem 2).

The method of the proofs uses techniques in Koul (1977) and in Louhichi (2000).

2 Results

Since the ranks $(R_{i\Delta})_{1\le i\le n}$ are translation invariant, the rank statistics defined in (2) cannot estimate the intercept parameter α in the model (1). We shall be interested in the sequel with the model described in (1) with no intercept parameter α (another rank statistics for α based on residuals is defined in the literature and other assumptions will be set on the distribution function of the random error terms for its study).

Our main result is the following.

Theorem 1. *Suppose that, in the model (1) with no intercept parameter α, the errors $(\epsilon_i)_{1\le i\le n}$ are a sequence of strictly stationary associated random variables with marginal density f having a finite Fisher information. Suppose also that,*

$$\mathrm{Cov}(\epsilon_1, \epsilon_i) = \mathcal{O}(i^{-p}) \text{ for some } p > 4. \tag{7}$$

Let $\hat{\beta}$ be the R-estimator of β as defined in (3) with a bounded, nondecreasing and right-continuous *score function φ on $[0,1]$ such that*

$$\sum_{i=1}^{\infty} \mathrm{Cov}(\varphi(F(\epsilon_1)), \varphi(F(\epsilon_i))) < \infty, \tag{8}$$

$$b(\varphi, f) := \int_0^1 f(F^{-1}(t)) d\varphi(t) < \infty. \tag{9}$$

Define $\tau_n^2 = \mathrm{Var}\left(\sum_{i=1}^n (x_i - \bar{x})\varphi(F(\epsilon_i))\right)$, $a_x^2 = \sum_{i=1}^n (x_i - \bar{x})^2$ *where* $\bar{x} = n^{-1} \sum_{i=1}^n x_i$*. If*

$$0 < \liminf_{n\to\infty} \frac{\tau_n}{a_x} < \limsup_{n\to\infty} \frac{\tau_n}{a_x} < \infty, \ \ \limsup_{n\to\infty} \sqrt{n} \max_{1\le i\le n} |x_i|/a_x < \infty, \tag{10}$$

then

$$\frac{a_x^2}{\tau_n}(\hat{\beta} - \beta) \Longrightarrow \mathcal{N}(0, b^{-2}(\varphi, f)).$$

Comments

1. Suppose moreover that φ is A-Lipchitz i.e. $|\varphi(x) - \varphi(y)| \le A|x - y|$. Then condition (8) is immediate from (7). This follows from the following covariance inequality (known for associated random variables),
$$\mathrm{Cov}(\varphi(F(\epsilon_1)), \varphi(F(\epsilon_i))) \le \|f\|_\infty^2 A^2 \mathrm{Cov}(\epsilon_1, \epsilon_i). \tag{11}$$

2. If f is known, strongly unimodal (i.e $-\ln(f)$ is convex) with a finite Fisher information then an important role is assigned to the score function $\varphi(u) = \varphi(u, f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}$, $u \in]0,1[$ (which is nondecreasing). In fact with this choice of φ, the asymptotic variance of the R-estimator is
$$b^{-2}(\varphi, f) := \left(\int_0^1 f(F^{-1}(t)) d\varphi(t)\right)^{-2} = \left(-\int_0^1 \varphi(t) \frac{f'(F^{-1}(t))}{f(F^{-1}(t))} dt\right)^{-2} = I^{-2}(f).$$
For double-exponential (or respectively Logistic) density f that is for $f(x) = \frac{1}{2}e^{-|x|}$ (or respectively $f(x) = \frac{e^x}{(1+e^x)^2}$), the score function $\varphi(u, f)$ equals to $\mathrm{sign}(2u - 1)$ (respectively to $2u - 1$). In those cases, φ is bounded, nondecreasing and right-continuous. We note also that in both cases, thanks to some covariance inequalities based on (11), Condition (8) follows from (7).

3. We consider the following model already discussed in Section 4 of Koul (1977), for $1 \le i \le n$, $Z_i = \beta i + \epsilon_i$, where $(\epsilon_i)_{0\le i\le n}$ is a stationary Gaussian process satisfying $\mathrm{Cov}(\epsilon_0, \epsilon_i) = \rho^i$, for some $\rho \in]0,1[$. As it was proved in Koul's paper Conditions (10) are satisfied with $x_i = i$. The sequence $(\epsilon_i)_{i\ge 0}$ is associated since it is Gaussian with positive covariance. So Theorem 1 applies without Koul's additional assumption of the mixing coefficient $\alpha(j)$ (see Condition (4.2) there).

4. The foundation of the theory of R-estimation can be seen as dual to the theory of rank tests. In fact as in Hájek (1962), a study of the test of the hypothesis $\beta = 0$ against $\beta > 0$ is based on the linear rank statistics $S_x(\Delta)$: let $(X_i)_{1\le i\le n}$ be a sequence of random variables where X_i has density f_i. The null hypothesis H_0 is $f_1 = \cdots = f_n = f$. The alternative H_1 shall be $f_i(x) = f(x - \beta x_i)$ with $\beta > 0$ (the one-sided alternative, for instance), the x_i's are some known constants. If f is of logistic type (respectively of double-exponential type) the test uses a statistic

$S = \sum_{i=1}^n x_i R_i$ (respectively a statistics $S = \sum_{i=1}^n x_i \mathrm{sign}(R_i - \frac{1}{2}(n+1)))$ where R_i is the rank of X_i among $(X_i)_{1\le i\le n}$ (we refer to chapter 4 of Sidak et al. (1999) for this theory for independent data $(X_i)_{1\le i\le n}$)

3 Proofs

We first prove the result in the general case where $(\epsilon_i)_{1\le i\le n}$ is a stationary sequence of dependent random variables (cf. Theorem 2 below). In Subsection 3.5 we apply the general result to associated random variables $(\epsilon_i)_{1\le i\le n}$ and we prove Theorem 1.

3.1 The general case

The purpose of this subsection (cf. Theorem 2 below) is to study the asymptotic normality of $\hat\beta$, as defined in (3), for the model in (1) with $\alpha = 0$ and when the errors $(\epsilon_i)_{1\le i\le n}$ are a sequence of strictly stationary dependent random variables with continuous marginal distribution function F and absolutely continuous density f with finite Fisher information. Let $c = (c_{i,n})_{1\le i\le n}$ and $d = (d_{i,n})_{1\le i\le n}$ be sequences of real numbers such that:

$$\lim_{n\to\infty} \max_{1\le i\le n} |c_{i,n}| = 0, \quad \limsup_{n\to\infty} \sum_{i=1}^n c_{i,n}^2 < \infty, \sum_{i=1}^n d_{i,n} = 0, \quad \limsup_{n\to\infty} \sqrt{n} \max_{1\le i\le n} |d_{i,n}| < \infty$$
$$(c_{i,n} - c_{j,n})(d_{i,n} - d_{j,n}) \ge 0, \ \forall\ 1 \le i \le j \le n. \tag{12}$$

<u>Notations.</u> Define, for $t \in [0,1]$,

$$H_\delta(y) = \frac{1}{n}\sum_{i=1}^n F(y + \delta c_{i,n}), \ \ V_d(t,\delta) = \sum_{i=1}^n d_{i,n}\left(\mathbb{I}_{\epsilon_i \le H_\delta^{-1}(t)+\delta c_{i,n}} - F(H_\delta^{-1}(t) + \delta c_{i,n})\right)$$

Let $H_{n,\delta}$ be the empirical cumulative of $(\epsilon_j - c_{j,n}\delta)_{1\le j\le n}$ and $H_{n,\delta}^{-1}$ be the empirical quantile function defined, on $[0,1]$, by $H_{n,\delta}^{-1}(t) = \inf\{y \in \mathbb{R},\ H_{n,\delta}(y) \ge t\}$.

<u>Assumptions.</u> We suppose that the following limits are satisfied for any fixed $\delta \in \mathbb{R}$ and any $t \in [0,1]$.

(L) $\forall\ \epsilon > 0$, $\lim_{u\to 0} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{|t-s|\le u} |V_d(t,\delta) - V_d(s,\delta)| \ge \epsilon\right) = 0$,

(L') $\lim_{n\to\infty} \mathrm{Var}\left(\sum_{i=1}^n d_{i,n}\left(\mathbb{I}_{\epsilon_i \le F^{-1}(t)+\delta c_{i,n}} - \mathbb{I}_{\epsilon_i\le F^{-1}(t)}\right)\right) = 0$.

(L'') $\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,1]} \sqrt{n}|H_\delta(H_{n,\delta}^{-1}(t)) - t| \ge K\right) = 0$.

Theorem 2. *Suppose that, in the model (1) with $\alpha = 0$, the errors $(\epsilon_i)_{1\le i\le n}$ is a sequence of stationary dependent random variables with marginal distribution function F and density f having a finite Fisher information. Suppose that the limits (L), (L') and (L'') are satisfied for any sequences c and d fulfilling (12). Suppose that Conditions (10) hold. Let $\hat\beta$ be as defined in (3) with a bounded, nondecreasing and right-continuous score function φ. Then the following convergence in distribution, as n tends to infinity, are equivalent:*

$$\frac{1}{\tau_n}\sum_{i=1}^n (x_i - \bar{x})\varphi(F(\epsilon_i)) \Longrightarrow \mathcal{N}(0,1)$$

is equivalent to

$$\frac{a_x^2}{\tau_n}(\hat{\beta}-\beta) \Longrightarrow \mathcal{N}(0, b^{-2}(\varphi, f)),$$

where $b(\varphi, f)$*,* a_x^2 *and* τ_n *are as defined in Theorem 1.*

3.2 Proof of Theorem 2.

The following proposition is the main key for the proof of Theorem 2.

Proposition 1. *Let* $(\epsilon_i)_{1\leq i\leq n}$ *be a sequence of stationary dependent random variables with marginal distribution function* F *and absolutely continuous density* f *with a finite Fisher information. Suppose that the limits* (L)*,* (L') *and* (L'') *are satisfied. Define for bounded, nondecreasing and right-continuous function* φ *on* $[0,1]$*,* $S_d(\delta) = \sum_{i=1}^n d_{i,n}\varphi(\frac{R_{i\delta}(c)}{n+1})$*, where* $R_{i\delta}(c)$ *is the rank of* $\epsilon_i - \delta c_{i,n}$ *among* $(\epsilon_j - \delta c_{j,n})_{1\leq j\leq n}$ *for sequences* $(c_{i,n})$ *and* $(d_{i,n})$ *satisfying (12). Then for any* $\epsilon > 0$ *and for each* $0 < b < \infty$

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{|\delta|\leq b}\left|S_d(\delta) - S_d(0) + \delta\left(\sum_{i=1}^n c_{i,n}d_{i,n}\right)b(\varphi, f)\right| \geq \epsilon\right) = 0,$$

recall that $b(\varphi, f) = \int_0^1 f(F^{-1}(t))d\varphi(t)$*.*

We prove this proposition in the next section and we continue the proof of Theorem 2. Let $S_x(\Delta)$ be as in (2). Recall that $Z_i = \beta x_i + \epsilon_i$. Hence we can write $Z_i - \Delta x_i = \epsilon_i - \delta c_{i,n}$, $S_x(\Delta) = a_x S_d(\delta)$, with

$$d_{i,n} = \frac{x_i - \bar{x}}{a_x},\ \ c_{i,n} = \frac{x_i}{a_x},\ \ \delta = (\Delta - \beta)a_x,$$

where $\bar{x} = n^{-1}\sum_{i=1}^n x_i$ and $a_x^2 = \sum_{i=1}^n (x_i - \bar{x})^2$. Since $\limsup_{n\to\infty}\sqrt{n}\max_{1\leq i\leq n}|x_i|/a_x < \infty$, Conditions (12) are all satisfied. Our task is to apply Proposition 1 with those values of $d_{i,n}, c_{i,n}$ and δ. Proposition 1 gives for any $\epsilon > 0$, since $a_x^{-1}S_x(\beta) = S_d(0)$ and $\sum_{i=1}^n x_i(x_i - \bar{x}) = a_x^2$,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{|\delta|\leq b}\left|a_x^{-1}S_x(\beta + \delta/a_x) - a_x^{-1}S_x(\beta) + \delta b(\varphi, f)\right| \geq \epsilon\right) = 0.$$

We have then by (16) (cf. Lemma 1 below), for any $\epsilon > 0$,

$$\lim_{b\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{|\delta|\leq b}\left|a_x^{-1}S_x(\beta + \delta/a_x) - a_x^{-1}S_x(\beta) + \delta b(\varphi, f)\right| < \epsilon, \left|\frac{a_x^{-1}S_x(\beta)}{b(\varphi, f)}\right| < b\right) = 1. \quad (13)$$

Define the event $\mathcal{A}_b$,

$$\mathcal{A}_b = \left\{\sup_{|\delta|\leq b}\left|a_x^{-1}S_x(\beta + \delta/a_x) - a_x^{-1}S_x(\beta) + \delta b(\varphi, f)\right| < \epsilon, \left|\frac{a_x^{-1}S_x(\beta)}{b(\varphi, f)}\right| < b\right\}. \quad (14)$$

It holds, since the path of $S_x(\cdot)$ is a.s monotone and for ϵ sufficiently small, that on the event $\mathcal{A}_b$, any value δ_n of δ that minimizes $a_x^{-1}|S_x(\beta + \delta/a_x)|$ belongs necessarily to $[-b, b]$. Define $\tilde{\delta} = \frac{a_x^{-1}S_x(\beta)}{b(\varphi, f)}$, on the event $\mathcal{A}_b$, we have $a_x^{-1}|S_x(\beta + \tilde{\delta}/a_x)| \leq \epsilon$. We deduce by definition of δ_n that,

$$a_x^{-1}|S_x(\beta + \delta_n/a_x)| \leq \epsilon.$$

Consequently we have by (14) and on the event $\mathcal{A}_b$,

$$\left|\delta_n b(\varphi, f) - \frac{S_x(\beta)}{a_x}\right| \leq 2\epsilon.$$

Hence using the definitions of $\hat{\beta}$ and δ_n we obtain on $\mathcal{A}_b$,

$$\left|a_x(\hat{\beta} - \beta)b(\varphi, f) - \frac{S_x(\beta)}{a_x}\right| \leq 2\epsilon.$$

The last bound together with (13) and (14) gives,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|-a_x^{-1}S_x(\beta) + (\hat{\beta} - \beta)a_x b(\varphi, f)\right| < 2\epsilon\right) = 1. \tag{15}$$

Consequently $-a_x^{-1}S_x(\beta) + (\hat{\beta} - \beta)a_x b(\varphi, f)$ converges in probability to 0 as n tends to infinity. We now need the following lemma.

Lemma 1. *The quantity $a_x^{-1}|S_x(\beta) - \sum_{i=1}^n (x_i - \bar{x})\varphi(F(\epsilon_i))|$ converges in probability to 0 as n tends to infinity and*

$$\lim_{b\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\left|\frac{a_x^{-1}S_x(\beta)}{b(\varphi, f)}\right| \geq b\right) = 0. \tag{16}$$

We prove Lemma 1 in Subsection 3.4 and we continue the proof of Theorem 2. Lemma 1 together with (15) allows to deduce that $a_x|(\hat{\beta} - \beta) - b^{-1}(\varphi, f)a_x^{-2}\sum_{i=1}^n (x_i - \bar{x})\varphi(F(\epsilon_i))|$ converges in probability to 0 as n tends to infinity. Define $\tau_n^2 = \mathrm{Var}\left(\sum_{i=1}^n (x_i - \bar{x})\varphi(F(\epsilon_i))\right)$ and suppose that $\liminf_{n\to\infty} \frac{\tau_n}{a_x} > 0$. We have then,

$$\frac{a_x^2}{\tau_n}|(\hat{\beta} - \beta) - b^{-1}(\varphi, f)a_x^{-2}\sum_{i=1}^n (x_i - \bar{x})\varphi(F(\epsilon_i))| \Longrightarrow 0$$

in probability to 0 as n tends to infinity. The last convergence completes the proof of Theorem 2. □

3.3 Proof of Proposition 1

Define, for $t \in [0, 1]$, $S(t, \delta) = \sum_{i=1}^n d_{i,n}\mathbb{I}_{R_{i\delta}(c)\leq tn}$. We have, letting $t_n = (n+1)n^{-1}t$ and noting that $\sum_{i=1}^n d_{i,n} = 0$, $S((n+1)n^{-1}, \delta) = 0$, $S(0, \delta) = 0$,

$$S_d(\delta) = \sum_{i=1}^n d_{i,n}\varphi(\frac{R_{i\delta}(c)}{n+1}) = \int_0^1 \varphi(t)d(S(t_n, \delta)) = -\int_0^1 S(t_n, \delta)d\varphi(t).$$

Hence, using the fact that $S(t_n, \delta) = S(t_n \wedge 1, \delta)$, we get

$$\begin{aligned}
&S_d(\delta) - S_d(0) + \delta\left(\sum_{i=1}^n c_{i,n}d_{i,n}\right)b(\varphi, f)\\
&= -\int_0^1 [S(t_n, \delta) - S(t_n, 0) - \delta\left(\sum_{i=1}^n c_{i,n}d_{i,n}\right)f(F^{-1}(t))]d\varphi(t)\\
&= -\int_0^1 [S(t_n \wedge 1, \delta) - S(t_n \wedge 1, 0) - \delta\left(\sum_{i=1}^n c_{i,n}d_{i,n}\right)f(F^{-1}(t_n \wedge 1))]d\varphi(t)\\
&+\delta\left(\sum_{i=1}^n c_{i,n}d_{i,n}\right)\int_0^1 [f(F^{-1}(t)) - f(F^{-1}(t_n \wedge 1))]d\varphi(t).
\end{aligned}$$

The second term of the last equality tends to 0 as n tends to infinity by the continuity of $f(F^{-1}(\cdot))$ on $[0,1]$ (recall that $I(f)<\infty$), the boundedness of φ and the fact that $\limsup_{n\to\infty}|\sum_{i=1}^n c_{i,n}d_{i,n}|<\infty$ (which follows from Conditions (12)). The proof of the proposition is then complete if we check that for any $\epsilon>0$ and $b>0$,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,1],\,|\delta|\le b}\left|S(t,\delta)-S(t,0)-\delta\left(\sum_{i=1}^n c_{i,n}d_{i,n}\right)f(F^{-1}(t))\right|\ge\epsilon\right)=0. \tag{17}$$

For this suppose without loss of generality that the sequence $(-d_{i,n})_{1\le i\le n}$ is increasing, hence by (12) the sequence $(-c_{i,n})_{1\le i\le n}$ is also increasing and Corollary 2 of Lehmann (1966) (page 1150) allows to deduce that for any fixed t, the function $\delta\longmapsto S(t,\delta)$ is nondecreasing. From this property, and since δ runs over a compact set $[-b,b]$, one can choose a suitable finite partition $(\delta_j)_{1\le j\le r}$ of $[-b,b]$ in such a way that,

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[0,1],\,|\delta|\le b}\left|S(t,\delta)-S(t,0)-\delta\left(\sum_{i=1}^n c_{i,n}d_{i,n}\right)f(F^{-1}(t))\right|\ge\epsilon\right)\\&\le\sum_{j=1}^r\mathbb{P}\left(\sup_{t\in[0,1]}\left|S(t,\delta_j)-S(t,0)-\delta_j\left(\sum_{i=1}^n c_{i,n}d_{i,n}\right)f(F^{-1}(t))\right|\ge\epsilon/2\right).\end{aligned}$$

We have then to prove (instead of (17)) that for any $|\delta|\le b$,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,1]}\left|S(t,\delta)-S(t,0)-\delta\left(\sum_{i=1}^n c_{i,n}d_{i,n}\right)f(F^{-1}(t))\right|\ge\epsilon\right)=0. \tag{18}$$

The proof of (18) is along Lemma 2.3 in Koul (1977) and it is based on the following technical lemma (mainly (20) and (19) below). Recall first that for $t\in[0,1]$,

$$H_\delta(y)=\frac{1}{n}\sum_{i=1}^n F(y+\delta c_{i,n}),\ \ L_{i,n}(t,\delta)=F(H_\delta^{-1}(t)+\delta c_{i,n})$$

We have by using the uniform continuity of $f(F^{-1})$ and (12) (as it is done in Lemma 2.2 of Koul (1977))

$$\lim_{n\to\infty}\sup_{t\in[0,1]}\left|\sum_{i=1}^n d_{i,n}L_{i,n}(t,\delta)-\sum_{i=1}^n d_{i,n}L_{i,n}(t,0)-\delta\sum_{i=1}^n c_{i,n}d_{i,n}f(F^{-1}(t))\right|=0. \tag{19}$$

Lemma 2. *Suppose that the limits (L) and (L″) are satisfied. Then for any fixed $|\delta|\le b$, one has, for any $\epsilon>0$,*

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,1]}\left|\left(S(t,\delta)-\sum_{i=1}^n d_{i,n}L_{i,n}(t,\delta)\right)-\left(S(t,0)-\sum_{i=1}^n d_{i,n}L_{i,n}(t,0)\right)\right|\ge\epsilon\right)=0. \tag{20}$$

Proof of Lemma 2. Recall that $H_{n,\delta}$ is the empirical cumulative of $(\epsilon_j-c_{j,n}\delta)_{1\le j\le n}$ and $H_{n,\delta}^{-1}$ is the empirical quantile function, defined on $[0,1]$, by $H_{n,\delta}^{-1}(t)=\inf\{y\in\mathbb{R},\ H_{n,\delta}(y)\ge t\}$. We have, since the event $R_{i\delta}(c)\le tn$ is equivalent to $H_{n,\delta}(\epsilon_i-\delta c_{i,n})\le t$,

$$S(t,\delta)=\sum_{i=1}^n d_{i,n}\mathbb{I}_{H_{n,\delta}(\epsilon_i-\delta c_{i,n})\le t}$$

We write, since $(H_{n,\delta}(y) \le t)$ is equivalent to $(H^{-1}_{n,\delta}(t) \ge y)$ that

$$
\begin{aligned}
S(t,\delta) - \sum_{i=1}^n d_{i,n} L_{i,n}(t,\delta) &= \sum_{i=1}^n d_{i,n} \mathbb{I}_{\epsilon_i - \delta c_{i,n} \le H^{-1}_{n,\delta}(t)} - \sum_{i=1}^n d_{i,n} L_{i,n}(t,\delta) \qquad (21)\\
&= \sum_{i=1}^n d_{i,n} \mathbb{I}_{\epsilon_i \le H^{-1}_{n,\delta}(t) + \delta c_{i,n}} - \sum_{i=1}^n d_{i,n} L_{i,n}(t,\delta)\\
&= \sum_{i=1}^n d_{i,n} (\mathbb{I}_{\epsilon_i \le H^{-1}_{n,\delta}(t) + \delta c_{i,n}} - F(H^{-1}_{n,\delta}(t) + \delta c_{i,n})) + \sum_{i=1}^n d_{i,n} (F(H^{-1}_{n,\delta}(t) + \delta c_{i,n})) - L_{i,n}(t,\delta))\\
&= V_d(t,\delta) + \left(V_d(H_\delta(H^{-1}_{n,\delta}(t)),\delta) - V_d(t,\delta) \right) + \sum_{i=1}^n d_{i,n} \left(L_{i,n}(H_\delta(H^{-1}_{n,\delta}(t)),\delta) - L_{i,n}(t,\delta) \right),
\end{aligned}
$$

where

$$
\begin{aligned}
V_d(t,\delta) &= \sum_{i=1}^n d_{i,n} \left(\mathbb{I}_{\epsilon_i \le H^{-1}_\delta(t) + \delta c_{i,n}} - F(H^{-1}_\delta(t) + \delta c_{i,n}) \right)\\
V_d(H_\delta(H^{-1}_{n,\delta}(t)),\delta) &= \sum_{i=1}^n d_{i,n} (\mathbb{I}_{\epsilon_i \le H^{-1}_{n,\delta}(t) + \delta c_{i,n}} - F(H^{-1}_{n,\delta}(t) + \delta c_{i,n}))
\end{aligned}
$$

and

$$
L_{i,n}(H_\delta(H^{-1}_{n,\delta}(t)),\delta) = F(H^{-1}_{n,\delta}(t) + \delta c_{i,n}) \qquad (22)
$$

We have, for any positive K,

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,1]} |V_d(H_\delta(H^{-1}_{n,\delta}(t)),\delta) - V_d(t,\delta)| \ge \epsilon \right)\\
&\le \mathbb{P}\left(\sup_{t\in[0,1]} \sqrt{n} |H_\delta(H^{-1}_{n,\delta}(t)) - t| \ge K \right) + \mathbb{P}\left(\sup_{|t-s|\le K/\sqrt{n}} |V_d(s,\delta) - V_d(t,\delta)| \ge \epsilon/2 \right)
\end{aligned}
$$

The last bounds and the limit (L) together with (L'') give, for any $\epsilon > 0$,

$$
\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\in[0,1]} |V_d(H_\delta(H^{-1}_{n,\delta}(t)),\delta) - V_d(t,\delta)| \ge \epsilon \right) = 0. \qquad (23)
$$

We have, since $\sum_{i=1}^n d_{i,n} = 0$,

$$
\begin{aligned}
&\sum_{i=1}^n d_{i,n} \left(L_{i,n}(H_\delta(H^{-1}_{n,\delta}(t)),\delta) - L_{i,n}(t,\delta) \right)\\
&= \sum_{i=1}^n d_{i,n} \left(L_{i,n}(H_\delta(H^{-1}_{n,\delta}(t)),\delta) - L_{i,n}(t,\delta) - (H_\delta(H^{-1}_{n,\delta}(t)) - t) \right)
\end{aligned}
$$

We deduce from this since $\limsup_{n\to\infty} \sqrt{n} \max_{1\le i\le n} |d_{i,n}| < \text{Cst}$,

$$
\begin{aligned}
&|\sum_{i=1}^n d_{i,n} \left(L_{i,n}(H_\delta(H^{-1}_{n,\delta}(t)),\delta) - L_{i,n}(t,\delta) \right)| \qquad (24)\\
&\le \text{Cst}\sqrt{n} \max_{1\le i\le n} \sup_{t\in[0,1]} |L_{i,n}(H_\delta(H^{-1}_{n,\delta}(t)),\delta) - L_{i,n}(t,\delta) - (H_\delta(H^{-1}_{n,\delta}(t)) - t)|.
\end{aligned}
$$

Hence, for any $K > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,1]} |\sum_{i=1}^n d_{i,n}\left(L_{i,n}(H_\delta(H_{n,\delta}^{-1}(t)),\delta) - L_{i,n}(t,\delta)\right)| \geq \epsilon\right) \\
&\leq \mathbb{P}\left(\mathrm{Cst}\sqrt{n}\max_{1\leq i\leq n}\sup_{t\in[0,1]} |L_{i,n}(H_\delta(H_{n,\delta}^{-1}(t)),\delta) - L_{i,n}(t,\delta) - (H_\delta(H_{n,\delta}^{-1}(t)) - t)| \geq \epsilon\right) \\
&\leq \mathbb{P}\left(\sqrt{n}\sup_{t\in[0,1]} |H_\delta(H_{n,\delta}^{-1}(t)) - t| \geq K\right) \\
&+\mathbb{P}\left(\max_{1\leq i\leq n}\sup_{|t-s|\leq Kn^{-1/2}} \sqrt{n}|L_{i,n}(t,\delta) - L_{i,n}(s,\delta) - (t-s)| \geq \epsilon\mathrm{Cst}\right).
\end{aligned}$$

We have, by arguing as (2.9) in Koul (1977) that $\lim_{n\to\infty}\max_{1\leq i\leq n}\sup_{t\in[0,1]}\frac{\partial}{\partial t}L_{i,n}(t,\delta) = 1$ and

$$\lim_{n\to\infty}\max_{1\leq i\leq n}\sup_{|t-s|\leq Kn^{-1/2}} \sqrt{n}|L_{i,n}(t,\delta) - L_{i,n}(s,\delta) - (t-s)| = 0,$$

consequently, by using (L''), we obtain

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,1]} |\sum_{i=1}^n d_{i,n}\left(L_{i,n}(H_\delta(H_{n,\delta}^{-1}(t)),\delta) - L_{i,n}(t,\delta)\right)| \geq \epsilon\right) = 0. \tag{25}$$

We get, by collecting (21), (23), (24) and (25): for any fixed $|\delta| \leq b$, one has, for any $\epsilon > 0$,

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,1]}\left|S(t,\delta) - \sum_{i=1}^n d_{i,n}L_{i,n}(t,\delta) - V_d(t,\delta)\right| \geq \epsilon\right) = 0, \tag{26}$$

The limit (20) follows by using the limit (26) together with the following limit (which proof is along the lines of that of Lemma 1.1 in Koul (1977) and it uses the limits (L) and (L')),

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,1]} |V_d(t,\delta) - V_d(t,0)| \geq \epsilon\right) = 0. \tag{27}$$

□

3.4 Proof of Lemma 1.

Recall that the linear statistics defined in (2) when Δ equals the true value β is, (recall that $S(t_n,0) = \sum_{i=1}^n d_{i,n}\mathbb{I}_{R_{i\beta}\leq nt_n}$, $d_{i,n} = a_x^{-1}(x_i - \bar{x})$),

$$S_x(\beta) = \sum_{i=1}^n (x_i - \bar{x})\varphi(\frac{R_{i\beta}}{n+1}),$$

where $R_{i\beta}$ is the rank of $Z_i - \beta x_i = \epsilon_i$ among $(\epsilon_j)_{1\leq j\leq n}$. We have almost surely, letting $t_n = (n+1)n^{-1}t$ and noting that $\sum_{i=1}^n d_{i,n} = 0$,

$$\begin{aligned}
a_x^{-1}S_x(\beta) = \sum_{i=1}^n d_{i,n}\varphi(\frac{R_{i\beta}}{n+1}) &= \int_0^1 \varphi(t)dS(t_n,0) = -\int_0^1 S(t_n,0)d\varphi(t) \\
\sum_{i=1}^n d_{i,n}\varphi(F(\epsilon_i)) &= \int_0^1 \varphi(t)d(\sum_{i=1}^n d_{i,n}\mathbb{I}_{\epsilon_i\leq F^{-1}(t)}) = -\sum_{i=1}^n d_{i,n}\int_0^1 \mathbb{I}_{\epsilon_i\leq F^{-1}(t)}d\varphi(t).
\end{aligned}$$

Consequently,

$$
\begin{aligned}
& a_x^{-1} S_x(\beta) - a_x^{-1} \sum_{i=1}^{n} (x_i - \bar{x}) \varphi(F(\epsilon_i)) \\
& = -\int_0^1 \left(\sum_{i=1}^{n} d_{i,n} \mathbb{1}_{Rank(\epsilon_i) \leq t_n n} - \sum_{i=1}^{n} d_{i,n} \mathbb{1}_{Rank(\epsilon_i) \leq tn} \right) d\varphi(t) \\
& + \int_0^1 \left(\sum_{i=1}^{n} d_{i,n} \mathbb{1}_{\epsilon_i \leq F^{-1}(t)} - \sum_{i=1}^{n} d_{i,n} \mathbb{1}_{Rank(\epsilon_i) \leq tn} \right) d\varphi(t).
\end{aligned}
$$

Hence (noting that $\sum_{i=1}^{n} \mathbb{1}_{tn < Rank(\epsilon_i) \leq t(n+1)} \leq 1$),

$$
\begin{aligned}
& \left| a_x^{-1} S_x(\beta) - a_x^{-1} \sum_{i=1}^{n} (x_i - \bar{x}) \varphi(F(\epsilon_i)) \right| \\
& \leq \sum_{i=1}^{n} |d_{i,n}| \int_0^1 \mathbb{1}_{tn < Rank(\epsilon_i) \leq (n+1)t} d\varphi(t) + \int_0^1 \left| \sum_{i=1}^{n} d_{i,n} \mathbb{1}_{\epsilon_i \leq F^{-1}(t)} - \sum_{i=1}^{n} d_{i,n} \mathbb{1}_{Rank(\epsilon_i) \leq tn} \right| d\varphi(t) \\
& \leq \max_{1 \leq i \leq n} |d_{i,n}| (\varphi(1) - \varphi(0)) + \sup_{t \in [0,1]} | \sum_{i=1}^{n} d_{i,n} \mathbb{1}_{Rank(\epsilon_i) \leq tn} - \sum_{i=1}^{n} d_{i,n} \mathbb{1}_{\epsilon_i \leq F^{-1}(t)} | (\varphi(1) - \varphi(0)).
\end{aligned}
\tag{28}
$$

Now (26) applied with $\delta = 0$ allows to deduce that

$$
\sup_{t \in [0,1]} | \sum_{i=1}^{n} d_{i,n} \mathbb{1}_{Rank(\epsilon_i) \leq tn} - \sum_{i=1}^{n} d_{i,n} \mathbb{1}_{\epsilon_i \leq F^{-1}(t)} |
$$

converges in probability to 0 as n tends to infinity. This fact together with (28) proves the first part of Lemma 1. Let us now prove (16). We have,

$$
\begin{aligned}
& \mathbb{P} \left(\left| \frac{a_x^{-1} S_x(\beta)}{b(\varphi, f)} \right| \geq b \right) \leq \mathbb{P} \left(\left| a_x^{-1} S_x(\beta) - a_x^{-1} \sum_{i=1}^{n} (x_i - \bar{x}) \varphi(F(\epsilon_i)) \right| \geq b(\varphi, f) b/2 \right) \\
& + \mathbb{P} \left(\left| a_x^{-1} \sum_{i=1}^{n} (x_i - \bar{x}) \varphi(F(\epsilon_i)) \right| \geq b(\varphi, f) b/2 \right).
\end{aligned}
$$

We deduce, thanks to the first part of this lemma and to Markov Inequality,

$$
\begin{aligned}
& \limsup_{n \to \infty} \mathbb{P} \left(\left| \frac{a_x^{-1} S_x(\beta)}{b(\varphi, f)} \right| \geq b \right) \\
& \leq \limsup_{n \to \infty} \mathbb{P} \left(\left| a_x^{-1} \sum_{i=1}^{n} (x_i - \bar{x}) \varphi(F(\epsilon_i)) \right| \geq b(\varphi, f) b/2 \right) \\
& \leq \frac{4}{b^2(\varphi, f) b^2} \limsup_{n \to \infty} \frac{\tau_n^2}{a_x^2}
\end{aligned}
$$

Consequently (16) holds by letting $b \to \infty$ in the last inequality. □

3.5 Proof of Theorem 1

We shall give conditions under which the limits (L), (L') and (L'') are satisfied for associated random variables.

Proposition 2. *Let $(\epsilon_i)_{1\le i\le n}$ be a sequence of associated random variables with identical continuous marginal distribution F and bounded density f. Let $c=(c_{i,n})_{1\le i\le n}$ and $d=(d_{i,n})_{1\le i\le n}$ be sequences of real numbers such that*

$$\lim_{n\to\infty}\max_{1\le i\le n}|c_{i,n}|=0,\ \ \limsup_{n\to\infty}\sqrt{n}\max_{1\le i\le n}|d_{i,n}|<\infty. \tag{29}$$

Suppose that there exists a real number $\rho\in]0,1[$ such that

$$\sup_{j\in\mathbb{N}}\sum_{i=1}^{\infty}\mathrm{Cov}^{\rho/3}(\epsilon_i,\epsilon_j)<\infty.$$

Then, for any $\delta\in\mathbb{R}$,

$$\lim_{n\to\infty}\mathrm{Var}\left(\sum_{i=1}^{n}d_{i,n}\left(\mathbb{I}_{\epsilon_i\le F^{-1}(t)+\delta c_{i,n}}-\mathbb{I}_{\epsilon_i\le F^{-1}(t)}\right)\right)=0.$$

Proof of Proposition 2. We suppose without loss of generality that $\delta c_{i,n}\ge 0$ and $\delta c_{j,n}\ge 0$. Then $\mathbb{I}_{\epsilon_i\le F^{-1}(t)+\delta c_{i,n}}-\mathbb{I}_{\epsilon_i\le F^{-1}(t)}=\mathbb{I}_{F^{-1}(t)<\epsilon_i\le F^{-1}(t)+\delta c_{i,n}}=\mathbb{I}_{t<F(\epsilon_i)\le F(F^{-1}(t)+\delta c_{i,n})}$. We have to control

$$\left|\mathrm{Cov}(\mathbb{I}_{t<F(\epsilon_i)\le F(F^{-1}(t)+\delta c_{i,n})},\mathbb{I}_{t<F(\epsilon_j)\le F(F^{-1}(t)+\delta c_{j,n})})\right|$$

On one hand, Hölder's inequality gives, since $F(\epsilon_i)$ and $F(\epsilon_j)$ are uniformly distributed,

$$\begin{aligned}&\left|\mathrm{Cov}(\mathbb{I}_{t<F(\epsilon_i)\le F(F^{-1}(t)+\delta c_{i,n})},\mathbb{I}_{t<F(\epsilon_j)\le F(F^{-1}(t)+\delta c_{j,n})})\right|\\&\le|F(F^{-1}(t)+\delta c_{i,n})-t|^{1/2}|F(F^{-1}(t)+\delta c_j)-t|^{1/2}\\&\le\|f\|_\infty|\delta|\max_{1\le i\le n}|c_{i,n}|.\end{aligned}$$

On the other hand, Yu's inequality (1993) gives,

$$\begin{aligned}&\left|\mathrm{Cov}(\mathbb{I}_{t<F(\epsilon_i)\le F(F^{-1}(t)+\delta c_{i,n})},\mathbb{I}_{t<F(\epsilon_j)\le F(F^{-1}(t)+\delta c_j)})\right|\\&\le 16\mathrm{Cov}^{1/3}(F(\epsilon_i),F(\epsilon_j))\end{aligned}$$

Consequently, for any $\rho\in]0,1[$

$$\begin{aligned}&\left|\mathrm{Cov}(\mathbb{I}_{t<F(\epsilon_i)\le F(F^{-1}(t)+\delta c_{i,n})},\mathbb{I}_{t<F(\epsilon_j)\le F(F^{-1}(t)+\delta c_j)})\right|\\&\le 16^\rho\|f\|_\infty^{1-\rho}|\delta|^{1-\rho}\max_{1\le i\le n}|c_{i,n}|^{1-\rho}\mathrm{Cov}^{\rho/3}(F(\epsilon_i),F(\epsilon_j)),\end{aligned}$$

and, by using the fact that $\mathrm{Cov}(F(\epsilon_i),F(\epsilon_j))\le\|f\|_\infty^2\mathrm{Cov}(\epsilon_i,\epsilon_j)$,

$$\begin{aligned}&\mathrm{Var}\left(\sum_{i=1}^{n}d_{i,n}\left(\mathbb{I}_{\epsilon_i\le F^{-1}(t)+\delta c_{i,n}}-\mathbb{I}_{\epsilon_i\le F^{-1}(t)}\right)\right)\\&\le\mathrm{Cst}\max_{1\le i\le n}|c_{i,n}|^{1-\rho}\sum_{i=1}^{n}\sum_{j=1}^{n}|d_{i,n}||d_{j,n}|\mathrm{Cov}^{\rho/3}(F(\epsilon_i),F(\epsilon_j))\\&\le\mathrm{Cst}\|f\|_\infty^{2\rho/3}(n\max_{1\le i\le n}|d_{i,n}|^2)(\max_{1\le i\le n}|c_{i,n}|^{1-\rho})\sup_{j\in\mathbb{N}}\sum_{i=1}^{n}\mathrm{Cov}^{\rho/3}(\epsilon_i,\epsilon_j).\end{aligned}$$

The last inequality and the limits in (29) complete the proof of this proposition. □

We also need the following proposition whose proof is in Subsection 3.6 below.

Proposition 3. *Let $(\epsilon_i)_{1\le i\le n}$ be a sequence of strictly stationary associated random variables with continuous marginal distribution function F and bounded density f. Let $(d_{i,n})_{1\le i\le n}$ and $(c_{i,n})_{1\le i\le n}$ be two sequences of real numbers satisfying (29). Define, for $t\in[0,1]$,*

$$V_d(t,\delta)=\sum_{i=1}^{n} d_{i,n}\left(\mathbb{I}_{\epsilon_i\le H_\delta^{-1}(t)+\delta c_{i,n}}-F(H_\delta^{-1}(t)+\delta c_{i,n})\right)$$

If $\mathrm{Cov}(\epsilon_1,\epsilon_n)=\mathcal{O}(n^{-p})$, *for some $p>4$, then for every $\epsilon>0$*

$$\lim_{u\to 0}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{|t-s|\le u}|V_d(t,\delta)-V_d(s,\delta)|\ge\epsilon\right)=0.$$

Proposition 4. *Let $(\epsilon_i)_{1\le i\le n}$ be a sequence of strictly stationary associated random variables with continuous marginal distribution function F and bounded density f. Let $(d_{i,n})_{1\le i\le n}$ and $(c_{i,n})_{1\le i\le n}$ be two sequences of real numbers satisfying (29). If* $\mathrm{Cov}(\epsilon_1,\epsilon_n)=\mathcal{O}(n^{-p})$, *for some $p>4$, then one has,*

$$\lim_{K\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sqrt{n}\sup_{t\in[0,1]}|H_{n,\delta}(H_\delta^{-1}(t))-t|\ge K\right)=0 \qquad (30)$$

$$\lim_{K\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sqrt{n}\sup_{t\in[0,1]}|H_{\delta}(H_{n,\delta}^{-1}(t))-t|\ge K\right)=0. \qquad (31)$$

Proof of Proposition 4. We have (recall that $H_{n,\delta}(t)=\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\epsilon_i-\delta c_{i,n}\le t}$, $H_\delta(y)=\frac{1}{n}\sum_{i=1}^{n}F(y+\delta c_{i,n})$),

$$V_1(t,\delta):=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\mathbb{1}_{\epsilon_i\le H_\delta^{-1}(t)+\delta c_{i,n}}-F(H_\delta^{-1}(t)+\delta c_{i,n})\right)=\sqrt{n}\left(H_{n,\delta}(H_\delta^{-1}(t))-H_\delta(H_\delta^{-1}(t))\right).$$

Let t and t_0 be fixed in $[0,1]$. We have, for any $m\in\mathbb{N}^*$

$$|V_1(t,\delta)|\le|V_1(t_0,\delta)|+m\sup_{|s-s'|\le 1/m}|V_1(s,\delta)-V_1(s',\delta)|$$

Consequently, for any $m>1$,

$$\mathbb{P}\left(\sup_{t\in[0,1]}|V_1(t,\delta)|\ge m\right)\le\mathbb{P}\left(|V_1(t_0,\delta)|\ge m/2\right)+\mathbb{P}\left(\sup_{|s-s'|\le\frac{1}{m}}|V_1(s,\delta)-V_1(s',\delta)|\ge\frac{1}{2}\right).$$

Let us note that $V_1(t,\delta)$ is $V_d(t,\delta)$ with $d_{i,n}=n^{-1/2}$. Hence the last limit together with Proposition 3 and the same arguments as that used in the proof of Proposition 2, give,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{t\in[0,1]}|V_1(t,\delta)|\ge m\right)=0.$$

The first part of the proposition is then proved. Let us now prove the second part. We have,

$$\begin{aligned}
|H_\delta(H_{n,\delta}^{-1}(t)) - t| &\leq |H_\delta(H_{n,\delta}^{-1}(t)) - H_{n,\delta}(H_{n,\delta}^{-1}(t))| + |H_{n,\delta}(H_{n,\delta}^{-1}(t)) - t| \\
&\leq \sup_{x\in\mathbb{R}} |H_\delta(x) - H_{n,\delta}(x)| + \sup_{t\in[0,1]} |H_{n,\delta}(H_{n,\delta}^{-1}(t)) - t| \\
&= \sup_{t\in[0,1]} |H_{n,\delta}(H_\delta^{-1}(t)) - t| + \sup_{t\in[0,1]} |H_{n,\delta}(H_{n,\delta}^{-1}(t)) - t| \\
&\leq \sup_{t\in[0,1]} |H_{n,\delta}(H_\delta^{-1}(t)) - t| + \frac{1}{n}
\end{aligned}$$

Consequently,

$$\sqrt{n} \sup_{t\in[0,1]} |H_\delta(H_{n,\delta}^{-1}(t)) - t| \leq \sqrt{n} \sup_{t\in[0,1]} |H_{n,\delta}(H_\delta^{-1}(t)) - t| + \frac{1}{\sqrt{n}}.$$

The last inequality together with (30) proves (31). □

End of the proof of Theorem 1. Our task is to apply Theorem 2 to the associated sequences $(\varphi(F(\epsilon_i)))_i$ (associated since φ and F are both nondecreasing). Since the limits $(L), (L'), (L'')$ are satisfied thanks to Propositions 2, 3 and 4, we have only to prove the asymptotic normality of $\tau_n^{-1}\sum_{i=1}^n (x_i - \bar{x})\varphi(F(\epsilon_i))$. For this we apply Proposition 5 below with $(a_{i,n})_{1\leq i\leq n} = (\tau_n^{-1}(x_i - \bar{x}))_{1\leq i\leq n}$. The proof of this proposition is in the spirit of Theorem 2.3 of Peligrad and Utev (1997). The main difference is that in Proposition 5 the coefficients $(a_{i,n})_{1\leq i\leq n}$ are not assumed to be positive.

Proposition 5. *Let $(\xi_i)_{1\leq i\leq n}$ be a sequence of stationary associated and bounded random variables. Let $(a_{i,n})_{1\leq i\leq n}$ be a triangular arrays of real numbers such that $\sum_{i=1}^n a_{i,n} = 0$. Suppose that* $\mathrm{Var}(\sum_{i=1}^n a_{i,n}\xi_i) = 1$ *and that*

$$\lim_{n\to\infty} \max_{1\leq i\leq n} |a_{i,n}| = 0, \quad \sup_{n\in\mathbb{N}^*} \sum_{i=1}^n a_{i,n}^2 < \infty.$$

If $\sum_{j=1}^\infty \mathrm{Cov}(\xi_1, \xi_i) < \infty$ *then* $\sum_{i=1}^n a_{i,n}\xi_i \Longrightarrow \mathcal{N}(0,1)$.

□

3.6 Proof of Proposition 3.

The proof is based on that of Louhichi (2000) and the references therein. Define $U_i = F(\epsilon_i)$ and $L_{i,n}(t,\delta) = F(H_\delta^{-1}(t) + \delta c_{i,n})$. Clearly $(U_i)_{1\leq i\leq n}$ is a sequence of strictly stationary associated random variables with uniform marginal distribution and

$$V_d(t,\delta) = \sum_{i=1}^n d_{i,n}\left(\mathbb{I}_{U_i\leq L_{i,n}(t,\delta)} - L_{i,n}(t,\delta)\right).$$

Define for $k \in \mathbb{N}$, the covering set T_k of $[0,1]$ by,

$$T_k = \{p2^{-k},\ 0 \leq p \leq 2^k\}.$$

For each $t_k \in T_k$, define the step function f_{t_k} by,

$$f_{t_k}(x) = \mathbb{I}_{x<t_k-2^{-k}} - (2^k x - 2^k t_k)\mathbb{I}_{x\in[t_k-2^{-k},t_k]}.$$

Let

$$\mathcal{F}_k = \{f_{t_k},\ t_k \in T_k\}.$$

Clearly, for $t_k = 2^{-k}[L_{i,n}(t,\delta)2^k]$,

$$f_{t_k}(x) \le \mathbb{I}_{x\le L_{i,n}(t,\delta)} \le f_{t_k+2^{1-k}}(x) =: g_{t_k}(x),$$

and

$$l_{i,n}(x) \le d_{i,n}\mathbb{I}_{x\le L_{i,n}(t,\delta)} \le L_{i,n}(x)$$

where $L_{i,n} - l_{i,n} = |d_{i,n}|(g_{t_k} - f_{t_k})$. Hence,

$$\mathbb{E}|L_{i,n} - l_{i,n}|(U_1) \le 3 \times 2^{-k} \max_{1\le i\le n} |d_{i,n}|.$$

Define,

$$\underline{V}(l) = \sum_{i=1}^{n} (l_{i,n}(U_i) - \mathbb{E}((l_{i,n}(U_1)))\,,\ \ \overline{V}(L) = \sum_{i=1}^{n} (L_{i,n}(U_i) - \mathbb{E}((L_{i,n}(U_1)))\,.$$

We have, since $d_{i,n}\mathbb{I}_{x\le L_{i,n}(t,\delta)} \le L_{i,n}$,

$$\begin{aligned}
&V_d(t,\delta) - \underline{V}(l)\\
&\le \sum_{i=1}^{n}(L_{i,n}(U_i) - l_{i,n}(U_i)) - \mathbb{E}(L_{i,n}(U_1) - l_{i,n}(U_1)) + \mathbb{E}(L_{i,n}(U_i) - d_{i,n}\mathbb{I}_{U_i\le L_{i,n}(t,\delta)})\\
&\le |\overline{V}(L) - \underline{V}(l)| + 3\,n \max_{1\le i\le n} |d_{i,n}|2^{-k}\\
&\le |\overline{V}(L) - \underline{V}(l)| + C\sqrt{n}2^{-k}
\end{aligned}$$

and, since $l_{i,n}(x) - d_{i,n}\mathbb{I}_{x\le L_{i,n}(t,\delta)} \le 0$,

$$\underline{V}(l) - V_d(t,\delta) \le 3\,n \max_{1\le i\le n} |d_{i,n}|2^{-k} \le C\sqrt{n}2^{-k}.$$

Hence (we refer also to (A2) in Louhichi (2000) for a precise justification)

$$\sup_{t\in[0,1]} |V_d(t,\delta) - \underline{V}(l)| \le \max_{f_k\in\mathcal{F}_k} |\overline{V}(L) - \underline{V}(l)| + C\sqrt{n}2^{-k}$$

This gives for $r > 2$ (see also (A.4) of Louhichi (2000))

$$\|\sup_{t\in[0,1]} |V_d(t,\delta) - \underline{V}(l)|\|_r \le \mathcal{N}_k^{1/r} \max_{f_k\in\mathcal{F}_k} \|\overline{V}(L) - \underline{V}(l)\|_r + C\sqrt{n}2^{-k}, \tag{32}$$

where $\mathcal{N}_k$ is the bracketing number defined as the smallest value of N for which there exist functions $f_1, \cdots, f_N$ in $\mathcal{F}_k$ such that for any $f \in \mathcal{F} = \{x \to \mathbb{I}_{x\le L_{i,n}(t,\delta)},\ t \in [0,1],\ i \le n\}$ there exists $f_u, f_v \in \mathcal{F}_k$ for which $f_u \le f \le f_v$ and $\mathbb{E}(f_v - f_u)(U_1) \le C2^{-k}$. For $\mathcal{F}_k$ as defined before

$$\mathcal{N}_k = \mathcal{O}(2^k),\ \text{ for any }\ k \in \mathbb{N}. \tag{33}$$

In order to control $\|\overline{V}(L) - \underline{V}(l)\|_r$ we need the following lemma whose proof is a straightforward generalization of (4.7) of Theorem 4.2 of Shao and Yu (1996).

Lemma 3. *Let $r > 2$ and f be a real valued function bounded by 1 with bounded first derivative. Let $(\epsilon_{i,})_{1\le i\le n}$ and $(d_{i,n})_{1\le i\le n}$ be as defined in Proposition 3. If,*

$$\mathrm{Cov}(\epsilon_1, \epsilon_n) = \mathcal{O}(n^{-p}), \text{ for some } p > r-1,$$

then for any positive μ there exists some positive constant k_μ independent of the function f for which

$$\mathbb{E}\left|\sum_{i=1}^{n} d_{i,n}(f(\epsilon_i) - \mathbb{E}(f(\epsilon_i))\right|^r$$
$$\le k_\mu \left\{ n^{1+\mu}\|f'\|_\infty^2 \max_{1\le i\le n} |d_{i,n}|^r + \max_{1\le i\le n} |d_{i,n}|^r \left(n \max_{i\le n} \sum_{j=1}^{n} |\mathrm{Cov}(f(\epsilon_i), f(\epsilon_j)| \right)^{r/2} \right\}$$

This moment inequality, together with Lemma 2 in Louhichi (2000) gives (for some constant C_r independent of n)

$$\|\overline{V}(L) - \underline{V}(l)\|_r \le C_r \left(2^{2k/r} n^{(1+\mu)/r} n^{-1/2} + 2^{-k/4}\right),$$

and consequently by (32),

$$\|\sup_{t\in[0,1]} |V_d(t,\delta) - \underline{V}(l)|\|_r \le \mathcal{N}_k^{1/r} C_r \left(2^{2k/r} n^{(1+\mu)/r} n^{-1/2} + 2^{-k/4}\right) + C\sqrt{n}2^{-k}$$

Let $k = k(n)$ be such that $\lim_{n\to\infty} \sqrt{n}2^{-k} = 0$, with this values of k we denote $\underline{V}(l(k(n))$ instead of $\underline{V}(l)$ (recall that $\underline{V}(l(k(n)) = \sum_{i=1}^{n} \min(d_{i,n}f_{t_k}, d_{i,n}g_{t_k})(U_i) - \mathbb{E}(\min(d_{i,n}f_k, d_{i,n}g_k)(U_1)))$. The last bound together with (33) gives, since $r > 5$,

$$\lim_{n\to\infty} \|\sup_{t\in[0,1]} |V_d(t,\delta) - \underline{V}(l(k(n)))|\,\|_r = 0. \tag{34}$$

Since $g_{t_k} = f_{t_k+2^{1-k}}$, $\underline{V}(l(k(n))$ is constructed from f_{t_k}, we will construct inductively $\underline{V}(l(j))$ as follows: for $f_j \in \mathcal{F}_j$, let $f_{j-1} \in \mathcal{F}_{j-1}$ that approximates the function f_j in the sense that

$$\mathbb{E}|f_j - f_{j-1}|(U_1) \le C2^{-j+1}.$$

Consequently, by arguing as (A6) and (A7) in Louhichi (2000), for any $m \le k(n)$,

$$\|\sup_{t\in[0,1]} |\underline{V}(l(k(n))) - \underline{V}(l(m))|\|_r$$
$$\le \sum_{j=m+1}^{k(n)} \|\sup_{t\in[0,1]} |\underline{V}(l(j)) - \underline{V}(l(j-1))|\|_r$$
$$\le \sum_{j=m+1}^{k(n)} \|\max_{f_j\in\mathcal{F}_j} |\underline{V}(l(j)) - \underline{V}(l(j-1))|\|_r$$
$$\le \sum_{j=m+1}^{k(n)} \mathcal{N}_j^{1/r} \max_{f_j\in\mathcal{F}_j} \|\underline{V}(l(j)) - \underline{V}(l(j-1))\|_r$$
$$\le C_r \sum_{j=m+1}^{k(n)} \mathcal{N}_j^{1/r} C_r \left(2^{2j/r} n^{(1+\mu)/r} n^{-1/2} + 2^{-j/4}\right).$$

We deduce, since $\mathcal{N}_j = \mathcal{O}(2^j)$, that for any fixed fixed $\epsilon > 0$, there exists m such that

$$\limsup_{n\to\infty} \| \sup_{t\in[0,1]} |\underline{V}(l(k(n))) - \underline{V}(l(m))| \|_r \le \epsilon.$$

The last bound together with (34) gives : for any fixed fixed $\epsilon > 0$, there exists m such that

$$\limsup_{n\to\infty} \| \sup_{t\in[0,1]} |V_d(t,\delta) - \underline{V}(l(m))| \|_r \le \epsilon.$$

Now, we argue exactly as in Andrews and Pollard (see their paragraph "comparison of pairs"), we obtain,

$$\| \sup_{|t-s|\le\delta} |V_d(t,\delta) - V_d(s,\delta)| \|_r \le 8\epsilon + \mathcal{N}_m^{2/r} \sup_{|t-s|\le\delta} \|V_d(t,\delta) - V_d(s,\delta)\|_r. \tag{35}$$

We have now to control $\|V_d(t,\delta) - V_d(s,\delta)\|_r$. Define

$$z_{i,n} = d_{i,n}(\mathbb{I}_{L_{i,n}(s,\delta)<U_i\le L_{i,n}(t,\delta)} - (L_{i,n}(t,\delta) - L_{i,n}(s,\delta)))$$

We argue as for (5.27) in Shao and Yu (1996). Let f_1 and f_2 be two regular functions such that, for $l \in \{1,2\}$,

$$\begin{aligned} &\|f_l'\|_\infty \le a^{-1},\ f_2(x) \le \mathbb{I}_{L_{i,n}(s,\delta)<x\le L_{i,n}(t,\delta)} \le f_1(x) \\ &0 \le f_1(x) - f_2(x) \le \mathbb{I}_{L_{i,n}(s,\delta)-a<x\le L_{i,n}(s,\delta)+a} + \mathbb{I}_{L_{i,n}(t,\delta)-a<x\le L_{i,n}(t,\delta)+a}, \end{aligned}$$

in such a way that

$$\mathbb{E}|f_l(U_n) - \mathbb{E}(f_l(U_n)) - \mathbb{I}_{L_{i,n}(s,\delta)<U_i\le L_{i,n}(t,\delta)} + (L_{i,n}(t,\delta) - L_{i,n}(s,\delta))| \le 8a.$$

Hence

$$|V_d(t,\delta) - V_d(s,\delta)| \le 8na \max_{1\le j\le n} |d_{j,n}| + \left| \sum_{i=1}^n d_{i,n}(f_1(U_i) - \mathbb{E}(f_1(U_i))) \right|.$$

From this inequality, we deduce as for (5.27) in Shao and Yu (1996) and with a suitable choice of a,

$$\begin{aligned} &\|V_d(t,\delta) - V_d(s,\delta)\|_r \le Cn^{1/2} \max_{1\le j\le n} |d_{j,n}| \times \\ &\left(n^{-(r-4-2\eta)/2(r+2)} + (\max_{1\le i\le n} \sum_{j=1}^\infty |\mathrm{Cov}(\mathbb{1}_{L_{i,n}(s,\delta)\le U_i\le L_{i,n}(t,\delta)}, \mathbb{1}_{L_{j,n}(s,\delta)\le U_j\le L_{j,n}(t,\delta)})|)^{1/2} \right). \end{aligned}$$

We have, by arguing as for the proofs of Lemmas 2 and 3 in Louhichi (2000),

$$\sum_{j=1}^\infty |\mathrm{Cov}(\mathbb{1}_{L_{i,n}(s,\delta)\le U_i\le L_{i,n}(t,\delta)}, \mathbb{1}_{L_{j,n}(s,\delta)\le U_j\le L_{j,n}(t,\delta)})| \le C \left(\max_{1\le j\le n} |L_{j,n}(t,\delta) - L_{j,n}(s,\delta)| \right)^{1/2}.$$

The two last bounds together with (35) and the fact that

$$\limsup_{n\to\infty} \max_{1\le j\le n} \frac{|L_{j,n}(t,\delta) - L_{j,n}(s,\delta)|}{|t-s|} \le 1$$

(see (2.9) in Koul (1977)), give

$$\|\sup_{|t-s|\leq u}|V_d(t,\delta)-V_d(s,\delta)|\|_r \leq 8\epsilon + C\mathcal{N}_m^{2/r}(n^{-(r-4-2\mu)/2(r+2)}+u^{1/4}).$$

Therefore, we obtain since $r > 4+2\mu$ and taking the limit in the last inequality,

$$\limsup_{n\to\infty}\|\sup_{|t-s|\leq u}|V_d(t,\delta)-V_d(s,\delta)|\|_r \leq \epsilon + C\mathcal{N}_m^{2/r}u^{1/4}.$$

Since m is fixed, we conclude that

$$\lim_{u\to 0}\limsup_{n\to\infty}\|\sup_{|t-s|\leq u}|V_d(t,\delta)-V_d(s,\delta)|\,\|_r = 0.$$

This completes the proof of Proposition 3 thanks to Markov inequality. □

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